Math 20 Spring 2013

Discrete Probability

Midterm Exam

Thursday April 25, 5:00–7:00 PM

Your name (please print): _____

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit. Partial credit will be awarded for serious progress toward a solution, but not for just writing something down. In addition, partial credit will be awarded for interesting ideas, even if they don't end up working. Bonus points (above full credit) may be awarded for especially interesting and unexpected solutions, at the instructor's discretion.

It is okay to leave your answers in unsimplified form. That is, if your answer is the sum or product of 5 numbers, you do not need to add or multiply them. Answers left in terms of binomial coefficients or factorials are also acceptable. However, do not leave any infinite sums or products, or sums or products of a variable number of terms.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

_____

For grader use only:

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. Suppose A, B, and C are independent events, and that $P(A) = 1/3$, $P(B) = 2/5$, and $P(C) = 3/4$.

   (a) What is $P(A \cup C)$?

   Solution: A and C are independent, so

   $$P(A \cup C) = P(A) + P(C) - P(A \cap C) = \frac{1}{3} + \frac{3}{4} - \frac{1}{3} \times \frac{3}{4} = \frac{5}{6}.$$

   (b) What is $P(A \cap B^c)$? (Recall that B^c denotes the complement of B.)

   Solution: We have $P(B^c) = 1 - P(B) = \frac{3}{5}$, and then by independence we have

   $$P(A \cap B^c) = P(A)P(B^c) = \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}.$$

   (c) What is $P(A \cup (B \cap C))$?

   Solution: We first work out

   $$P(B \cap C) = P(B)P(C) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}.$$

   Then

   $$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C) = \frac{1}{3} + \frac{3}{10} - \frac{1}{3} \times \frac{3}{10} = \frac{8}{15}.$$

2. Suppose you roll three dice. Compute the following:

(a) The expected value of the sum of the rolls.

Solution: Let X_1, X_2, X_3 be the rolls. We have $\mathbb{E}(X_i) = \frac{7}{2}$ for $i = 1, 2, 3$. Then

$$\mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 3 \times \frac{7}{2} = \frac{21}{2}.$$

(b) The expected value of the product of the rolls.

Solution: Since the rolls are independent, we have

$$\mathbb{E}(X_1 X_2 X_3) = \mathbb{E}(X_1)\mathbb{E}(X_2)\mathbb{E}(X_3) = \left(\frac{7}{2}\right)^3 = \frac{343}{8}.$$

(c) The variance of the sum of the rolls.

Solution: Since the rolls are independent, we have

$$V(X_1 + X_2 + X_3) = V(X_1) + V(X_2) + V(X_3) = 3V(X_1).$$

To compute the variance of X_1, we compute

$$V(X_1) = \mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2 = \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Hence the variance of the sum is $\frac{35}{4}$.

3. (a) A typical page in a book contains one typo per page. What is the probability that there are exactly 8 typos in a given 10-page chapter?

Solution: This is a Poisson distribution. Since the expected number of typos on one page is 1, the expected number of typos in 10 pages is 10. Hence, we use the Poisson formula with $\lambda = 10$ to get

$$e^{-10}\frac{10^8}{8!}.$$

(b) Suppose that every page in the chapter contains exactly 3000 characters, and there is still an average of one typo per page. What is the probability that there are exactly 8 typos in the 10-page chapter?

Solution: This time, we use a binomial distribution, where a typo counts as a success. The probability of success is $\frac{1}{3000}$; we have 30000 trials, so the probability of getting 8 successes is

$$\binom{30000}{8}\left(\frac{1}{3000}\right)^8\left(\frac{2999}{3000}\right)^{29992}.$$

(c) Since calculators are not permitted on this exam, it will be hard to compare the results of parts (a) and (b) numerically. But do you expect them to be close or far away? Why?

Solution: We expect them to be close because the probability of success is small and the number of trials is large, so the Poisson distribution approximates the binomial distribution well. Indeed, to four decimal places, both are .1126.

4. What is the probability that a random integer from 1 to 10000 is divisible by exactly two of 4, 5, 6, and 7?

Solution: In order to reduce clutter, let us define

$$a_n = \frac{1}{10000} \left\lfloor \frac{10000}{n} \right\rfloor$$

to be the probability that a random integer from 1 to 10000 is divisible by n. We first want to count the double overlaps: $P_2 = a_{20} + a_{12} + a_{28} + a_{30} + a_{35} + a_{42}$. However, we wish for the numbers divisible by three of 4, 5, 6, and 7 to be counted 0 times; so far they are counted three times. (For example, a number divisible by 4, 5, and 6 is counted as being divisible by 4 and 5, 4 and 6, and 5 and 6.) So, we need to subtract three times the triple overlaps: define $P_3 = a_{60} + a_{140} + a_{84} + a_{210}$. Once we have subtracted three times the triple overlaps, we have to worry about the quadruple overlaps. They are counted in each of the terms in P_2, so 6 times. We subtract them 3 times in each of the 4 terms in P_3, so 12 times. Hence, we have counted them -6 times so far, and we want them to be counted 0 times. So, we must add 6 times the quadruple overlaps: $P_4 = a_{420}$. Thus the final answer is

$$P_2 - 3P_3 + 6P_4.$$

5. You have 10 pairs of socks (so 20 socks total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks, at random. What is the expected number of complete pairs you have left?

Solution: The easiest and cleanest way of solving this problem is to break up the expected number of pairs remaining into 10 smaller random variables, one for each pair. For $1 \leq i \leq 10$, let X_i be a random variable, which takes on the value 1 if both socks in pair i survive the washing machine, and 0 if at least one sock is eaten. Then the expected number of pairs remaining is the sum of the $\mathbb{E}(X_i)$'s, or $10\mathbb{E}(X_1)$. To compute $\mathbb{E}(X_1)$, we need only work out the probability that both socks in pair 1 survive. The probability that the first sock in the pair survives is $\frac{16}{20}$, and the probability that the second sock survives given that the first one does is $\frac{15}{19}$. Hence the probability that both survive is $\frac{16}{20} \times \frac{15}{19} = \frac{12}{19}$. Thus the expected number of pairs remaining is $10 \times \frac{12}{19} = \frac{120}{19}$.

6. An urn contains 12 red balls, 5 green balls, and 8 blue balls. You pick 6 balls at random (without replacement). What is the probability that you get 4 red balls, 1 green ball, and 1 blue ball?

Solution: There are 25 balls, and you pick 6; this can ben done in $\binom{25}{6}$ ways. We must pick 12 red balls, which can be done in $\binom{12}{4}$ ways; similarly, we must pick 1 green ball in one of $\binom{5}{1}$ ways, and 1 blue ball in one of $\binom{8}{1}$ ways. Hence the probability of getting 4 red balls, 1 green ball, and 1 blue ball is

$$\frac{1}{\binom{25}{6}}\binom{12}{4}\binom{5}{1}\binom{8}{1} = \frac{18}{161}.$$

7. You are working on a difficult passage from a new piece you are learning on the piano. You wish to play it correctly 4 times before calling it a day. If you have a probability of $\frac{2}{3}$ of playing it correctly on every attempt, and the attempts are independent (unfortunately!), what is the probability that you attempt to play it at least 8 times?

Solution: There are several ways of doing this problem. One way is to count the complement, which is the probability of getting it right 4 times in under 8 tries. The probability of getting it right 4 times in n tries is $\binom{n-1}{3} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{n-4}$. Hence the probability of getting it right in under 8 tries is

$$\sum_{n=4}^{7} \binom{n-1}{3} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{n-4}.$$

The desired probability is 1 minus the probability above, which is $\frac{379}{2187}$.

8. Recall that the Catalan numbers C_n count the number of paths from $(0,0)$ to (n,n) with steps one unit up and one unit to the right, which never go below the line $y = x$. Prove that

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \cdots + C_{n-1} C_1 + C_n C_0.$$

Solution: We think of paths from $(0,0)$ to $(n+1, n+1)$ as "words" of letters U and R, where U denotes a step up and R denotes a step to the right. We construct a bijection between the paths counted by C_{n+1} and the paths counted on the right side. Suppose we have a path from $(0,0)$ to $(n+1, n+1)$. We look for the first time after $(0,0)$ at which the path meets the diagonal line $y = x$; suppose this happens at the point (k, k). Then the subpath from $(0,0)$ to (k, k) begins with a U and ends with an R. If we delete these two letters, we are left with a path from $(0,0)$ to $(k-1, k-1)$. Since (k, k) was the first intersection with the diagonal, this new subpath does not ever cross the line $y = x$, and so it is a path counted by C_{k-1}. The remainder of the original path, which went from (k, k) to $(n+1, n+1)$, now goes from $(k-1, k-1)$ to (n, n). Patching these two paths together is then something counted by $C_{k-1} C_{n-k+1}$.

If we really want to show that this is a bijection, we ought to construct an inverse map, which takes some path counted by $C_{k-1} C_{n-k+1}$ and gives back a path counted by C_{n+1}. To do this, we simply create the path which starts with a U, then continues with the C_{k-1} path, then continues with an R, then concludes with the C_{n-k+1} path. These two maps are inverses, meaning that when we start with one path, apply one of these maps, then apply the other, we get back where we started. So there are the same number of objects counted by each side.

10

9. It is well-known that in Hanover in the Spring, it is either sunny or rainy every day, and the probability that the weather one day is the same as the weather the previous day is $2/3$. Suppose it was sunny on May 1st and May 10th. What is the probability that it was sunny on May 3rd?

Solution: This is a problem about conditional probability. In order to parse the condition of it being sunny on May 1st and May 10th, we simply declare that it is sunny on May 1st and make that an assumption in everything that follows. We then have

$$P(\text{sunny on May } 3^{\text{rd}} \mid \text{sunny on May } 10^{\text{th}}) = \frac{P(\text{sunny on May } 3^{\text{rd}} \text{ and May } 10^{\text{th}})}{P(\text{sunny on May } 10^{\text{th}})}.$$

We can also reinterpret the numerator as

$$P(\text{sunny on May } 10^{\text{th}} \mid \text{sunny on May } 3^{\text{rd}}) \times P(\text{sunny on May } 3^{\text{rd}}).$$

The probability that it is sunny on May 3rd is $\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}$. In order to compute the probability that it is sunny on May 10th given that it is sunny on May 3rd, we need to work out the probability that there are an even number of switches in 7 possible switches. It wouldn't be bad to work that out directly by enumerating all the cases, but we've seen before that this can be done using the binomial theorem: this probability is

$$\frac{1}{2}\left(\left(\frac{2}{3} + \frac{1}{3}\right)^7 + \left(\frac{2}{3} - \frac{1}{3}\right)^7\right) = \frac{1094}{2187}.$$

Hence the entire numerator is

$$\frac{1094}{2187} \times \frac{5}{9} = \frac{5470}{19683}.$$

The computation for the denominator is very similar; it comes out to

$$\frac{1}{2}\left(\left(\frac{2}{3} + \frac{1}{3}\right)^9 + \left(\frac{2}{3} - \frac{1}{3}\right)^9\right) = \frac{9842}{19683}.$$

Putting this all together, we get a final answer of

$$\frac{2735}{4921}.$$

10. Suppose you deal 13 cards from a standard deck of 52 cards. What is the probability that the distribution of suits is 5, 5, 2, 1? (That is, you have 5 cards of one suit, 5 cards of another, 2 of another, and 1 of the last.)

Solution: There are $\binom{52}{13}$ ways of picking 13 cards. Let us first suppose that we have an ordering of the suits, so that perhaps we get 5 clubs, 5 diamonds, 2 hearts, and 1 spade. That can happen in $\binom{13}{5}\binom{13}{5}\binom{13}{2}\binom{13}{1}$ ways. However, we can rearrange the suits in 24 ways. But two of them are equivalent, since we're choosing 5 of two different suits. Hence the probability of getting that distribution is

$$\frac{12}{\binom{52}{13}}\binom{13}{5}^2\binom{13}{2}\binom{13}{1}.$$