Math 20, Fall 2017

Edgar Costa

Week 2

Dartmouth College

Resources

- Tutorial: Thursdays 7-8:30 pm
- Office hours: M: 5:30 - 7+ pm, Th: 8 - 9 am, and by appointment
- Study groups: Sundays 3:30pm - 5:00pm
  Students can begin registering for Study Groups on Sunday, September 17th by accessing: `studygroups.dartmouth.edu` and logging in with their netID and password.

Birthday paradox

How many people do we need to have in a room to make it a favorable bet (probability of success greater than 1/2) that at least two people in the room will have the same birthday?

If there are n people in the room, what is the probability that all of them have different birthday?

$$\frac{365 \cdot 364 \cdot 363 \cdots (365 - (n-1))}{365 \cdot 365 \cdot 365 \cdots 365}$$

Challenge

Why not

$$\frac{n \text{ objects unordered, without repetition, from } 365}{n \text{ objects unordered, with repetition, from } 365} = \frac{\binom{365}{n}}{\binom{n+(365-1)}{365-1}} ??$$

Independent events

Definition

Two events A and B are independent if $P(A \cap B) = P(A) \cdot P(B)$

Example (non independent events)

We have a box with n_B blue balls and n_R red balls, $n_B + n_R = N$.

- A = {choosing a red ball on the first pick}
- B = {choosing a blue ball on the second pick, **without** replacement}
- $P(A) = n_R / N$
- $P(A \cap B) = \frac{n_R n_B}{N(N-1)}$
- $P(B) = P(B \cap A) + P(B \cap (\text{not } A)) = \frac{n_R n_B}{N(N-1)} + \frac{n_B(n_B - 1)}{N(N-1)} = \frac{n_B}{N}$

Draw picture!

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Draw picture!

Independent events

Independence allows us to break down complex events into simpler ones.

You toss a fair coin n times, what is the probability of getting heads k times?

X = number of heads in n tosses

$$P(X = k) = ?$$

Assuming that each toss independent, the sequence with k H's followed by $(n-k)$ T's has probability

$$P(H_1)P(H_2)\cdots P(H_k)P(T_1)\cdots P(T_{n-k}) = p^k(1-p)^{n-k}.$$

There are $\binom{n}{k}$ ways to rearrange the sequence above, all with same probability,

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

If two events A and B are not independent we write

$$P(A \text{ and } B) = P(A) \cdot P(B|A),$$

where $P(B|A)$ stands for the probability that B occurs, given that A occurs.

It is called a **conditional probability**, is read as "the probability of B, given A."

Example

You have two boxes, X and Y. The box X has 2 red balls and 5 blue balls, the ball Y has 3 balls of each. Assume a box is chosen at random and a ball is chosen at random from it.

- What is the probability of picking the red ball, given that we pickedthe box X?
- What is the probability of picking a red ball?
- What is the probability of picking the box X, given that we picked a red ball?

Problem

A die is rolled twice. What is the probability that the sum of the faces is greater than 7, given that

- the first outcome was a 4?
- the first outcome was greater than 3?
- the first outcome was a 1?
- the first outcome was less than 5?

More problems

1. Suppose a drug test is 99% sensitive and 99% specific. That is, the test will produce 99% true positive results for drug users and 99% true negative results for non-drug users. Suppose that 0.5% of people are users of the drug. What is the probability that a randomly selected individual with a positive test is a user?
2. The three machines account for different amounts of the factory output, namely 20%, 30%, and 50%. The fraction of defective items produced is this: for the first machine, 5%; for the second machine, 3%; for the third machine, 1%. If an item is chosen at random from the total output and is found to be defective, what is the probability that it was produced by the third machine?

Source: `https://en.wikipedia.org/wiki/Bayes%27_theorem`

Formalizing probability - Random Variables, Sample Spaces, Outcomes and Events

- We represent the outcome of the experiment by a capital Roman letter, such as X, called a **random variable**.
- The **sample space** of the experiment is the set of all possible outcomes. If the sample space is either finite or countably infinite, the random variable is said to be discrete.
- The elements of a sample space are called **outcomes**.
- A subset of the sample space is called an **event**.

Formalizing probability - Distribution Functions

- Let X be a **random variable** which denotes the value of the outcome of a certain experiment.
- Let Ω be the **sample space** of the experiment (i.e., the set of all possible values of X, or equivalently, the set of all possible outcomes of the experiment.)

A distribution function for X is a function $m_X : \Omega \to \mathbb{R}$ which satisfies

1. $m_X(\omega) \geq 0$, for all $\omega \in \Omega$, and
2. $\sum_{\omega \in \Omega} m_X(\omega) = 1$.

For any subset E of Ω, we define the probability of E to be the number $P(E)$ given by

$$P(E) = \sum_{\omega \in E} m_X(\omega).$$

Example

- X = random variable with n equally likely events
- $\Omega = \{x_1, \dots, x_n\}$
- $m_X(x_i) = 1/n$
- $P(E) = \sum_{\omega \in E} m_X(\omega) = \#E \frac{1}{n} = \frac{\#E}{\#\Omega} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$

Examples

Three people, A, B, and C, are running for the same office, and we assume that one and only one of them wins. Suppose that A and B have the same chance of winning, but that C has only 1/2 the chance of A or B. What is the probability to win for each of the three people?

- Rolling a die
- Tossing a coin
- Picking a ball
- Rolling two dice
- Sum of two rolls

Basic set operations

A and B two events

- the intersection of A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- the union of A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- the complement of A is the set $\overline{A} = \{x : x \in \Omega \text{ and } x \notin A\}$

Properties

The probabilities assigned to events by a distribution function on a sample space Ω satisfy the following properties:

1. $P(E) \geq 0$ for every $E \subset \Omega$.
2. $P(\Omega) = 1$.
3. If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$.
4. If A and B are disjoint subsets, then $P(A \cup B) = P(A) + P(B)$
5. $P(\overline{A}) = 1 - P(A)$ for every $A \subset \Omega$.

Can you prove these?

Inclusion-Exclusion Principle

Inclusion-Exclusion principle

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots$$

- What is the probability of drawing a king or a heart?
- Two cards are drawn at random. What is the probability of drawing a king or an ace?

The hat problem (GS: Example 3.12)

In a restaurant n hats are checked and they are hopelessly scrambled; what is the probability that no one gets his own hat back?

- What is the probability that the i-th person gets her hat back?
- What is the probability that the i-th and the j-th person get their hat back?
- What is the probability that at least someone gets their hat back?

$$\binom{n}{1}\frac{n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} \cdots + (-1)^{n-1}\binom{n}{n}\frac{1}{n!}$$

$$\xrightarrow[n\to+\infty]{} \frac{1}{e}$$