Math 20, Test 2, May 12, 2015

Instructions. Show your work and explain carefully, except for short-answer problems. Calculators and other electronic aids are not permitted. Answers involving the number e may be left in that form, as in "ae^b" or "$1 - ae^b$", with numerical values for a, b. The first problem is worth 12 points, the others 11 points each.

1. Which of the following are distribution functions? (hint: at least one is not)

   (a) $m(j) = \frac{1}{10}$ for j an integer, $0 \le j \le 10$.
   No, the sum of the m-values is $11/10 \neq 1$.

   (b) $m(j) = \binom{73}{j}\left(\frac{1}{4}\right)^j\left(\frac{3}{4}\right)^{73-j}$ for j an integer, $0 \le j \le 73$.
   Yes, this is the binomial distribution with $n = 73$.

   (c) $m(j) = j/3$ for j an integer, $1 \le j \le 2$.
   Yes, the m-values are $1/3$ and $2/3$, which are nonnegative and add to 1.

   (d) $m(j) = \frac{2^j}{j!}e^{-2}$ for j a non-negative integer.
   Yes, this is the Poisson distribution with $\lambda = 2$.

2. Let X_1, X_2 be independent, uniform random variables on $\{1, 2, \ldots, 10\}$. Find

$$P(\text{the maximum of } X_1, X_2 \text{ is at least } 5).$$

   It's easier to compute the complementary probability, which is the event that both X_1 and X_2 are ≤ 4. Since they're independent, this probability is $(4/10)^2$. So, the answer is $1 - (4/10)^2 = 0.84$.

3. The number of hours a light bulb lasts is assumed to follow a geometric distribution with mean 10,000 hours of use. What is the approximate probability it fails in the first 5,000 hours of use?

   The geometric distribution with parameter p is $m(k) = q^{k-1}p$, where k is a positive integer, $q = 1 - p$, and $0 < p < 1$. The mean of this distribution is $1/p$ (either remember this, or work it out). So, we have $p = 1/10,000$. The probability that it fails for some $k \le 5000$ is

$$\sum_{k=1}^{5000} q^{k-1}p.$$

This is the exact probability. To estimate it, we sum the geometric progression to get

$$\frac{q^{5000}-1}{q-1}p = 1 - q^{5000} = 1 - \left(1 - \frac{1}{10{,}000}\right)^{5000}.$$

Since $(1-1/n)^n$ tends to e^{-1} as n tends to infinity, the high power here, which is

$$\left(\left(1 - \frac{1}{10{,}000}\right)^{10{,}000}\right)^{1/2}$$

should be close to $\mathrm{e}^{-1/2}$. Thus, the probability is approximately $1 - \mathrm{e}^{-1/2}$. (The exact probability is $0.393485\ldots$, while $1 - \mathrm{e}^{-1/2} = 0.393469\ldots$.)

4. A good typist averages a single typo in 1000 words. A chapter being typed has 10,000 words. Assuming a Poisson distribution, what is the probability there are at most 2 typos in the chapter?

   The parameter λ in this Poisson distribution is the average number of typos in 10,000 words, which is 10. Thus, the probability is

$$\frac{10^0}{0!}\mathrm{e}^{-10} + \frac{10^1}{1!}\mathrm{e}^{-10} + \frac{10^2}{2!}\mathrm{e}^{-10} = 61\mathrm{e}^{-10} \quad (= 0.002769\ldots).$$

5. Suppose X_1, X_2 are independent random variables and c is a constant. If the statement is always true, write True, otherwise write False.

   (a) $E(X_1 + cX_2) = E(X_1) + cE(X_2)$.
   True. Expectation is linear, and one does not need that X_1, X_2 are independent.

   (b) $V(X_1 - X_2) = V(X_1) + V(X_2)$.
   True. If X_1, X_2 are independent, so are $X_1, -X_2$, so $V(X_1 - X_2) = V(X_1) + V(-X_2)$. We always have $V(cX) = c^2V(X)$, so applying this with $c = -1$, we have the statement.

6. Flip a fair coin 3 times and win j dollars if the longest streak of consecutive heads or tails has length j. What is the expected value of this game and what is the variance?

   Of the 8 possibilities for 3 coin flips, 2 have longest run 1 (HTH and THT), 2 have longest run 3, and the remaining 4 have longest run 2. So the expectation is

$$1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2,$$

   or \$2. The mean square is

$$1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{4} = 4.5,$$

   so, using the result that the variance is the mean square minus the square mean, the variance is $4.5 - 2^2 = .5$.

7. Prove that if X is a random variable, then $E(X^2) \geq E(X)^2$.

Solution 1. The definition of variance is $E((X-\mu)^2)$, where $\mu = E(X)$. Thus,

$$V(X) = \sum_x (x-\mu)^2 P(X = x).$$

The two factors in each summand are both nonnegative, so therefore the entire sum is nonnegative, i.e., $V(X) \geq 0$. We learned that $V(X) = E(X^2) - E(X)^2$. Thus, $E(X^2) - E(X)^2 \geq 0$, which is equivalent to the inequality we are asked to prove.

Solution 2. Several students tried to do this problem without using variance. It is possible, but a little tricky. Here's a proof using the Cauchy–Schwarz inequality. This asserts that

$$\left|\sum_{i=1}^n a_i b_i\right| \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{i=1}^n b_i^2\right)^{1/2}.$$

This easily generalizes to convergent sums. We apply this inequality with the numbers a_i running over $x\sqrt{P(X{=}x)}$ and the numbers b_i running over the numbers $\sqrt{P(X{=}x)}$. Thus,

$$\begin{aligned}|E(X)| = \left|\sum_x xP(X{=}x)\right| &= \left|\sum_x x\sqrt{P(X{=}x)} \cdot \sqrt{P(X{=}x)}\right| \\ &\leq \left(\sum_x x^2 P(X{=}x)\right)^{1/2} \left(\sum_x P(X{=}x)\right)^{1/2}.\end{aligned}$$

The first sum just above is $E(X^2)$ and the second sum is 1. So, squaring the inequality, we get $E(X)^2 \leq E(X^2)$.

8. If $m_1(x)$ is uniform on $\{1, 2, \ldots, 6\}$ and $m_2(x)$ is uniform on $\{1, 2, \ldots, 10\}$, find

$$(m_1 * m_2)(8).$$

We learned about convolution in the context of adding two random variables. This problem can be thought of as rolling two dice, one with 6 faces with numbers 1 to 6 and the other with 10 faces with numbers 1 to 10. We are asked for the probability of rolling 8. Each choice of i on the first die and j on the second die has probability $\frac{1}{6} \cdot \frac{1}{10} = \frac{1}{60}$. There are 6 ways to make 8, namely $i + (8 - i)$ for $i = 1, 2, \ldots, 6$. So, the answer is $6 \cdot \frac{1}{60} = \frac{1}{10}$.

9. What does the Law of Large Numbers have to say about the probability of getting at least 75 heads when flipping a fair coin 100 times?

The Law of Large Numbers is both a qualitative statement and a quantitative statement. It says that if you add identically distributed independent random variables $X_1, \ldots, X_n$, each with mean μ and variance σ^2, then for each $\epsilon > 0$,

$$P\left(|A_n - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2},$$

where $A_n = S_n/n = (X_1 + \cdots + X_n)/n$. Thus, as $n \to \infty$, this probability approaches 0. Specifically in our case we have $n = 100$, $\sigma^2 = \frac{1}{4}$, and $\epsilon = \frac{25}{100} = \frac{1}{4}$. Thus,

$$P\left(\left|A_{100} - \frac{1}{2}\right| \geq \frac{1}{4}\right) \leq \frac{1/4}{100(1/4)^2} = \frac{4}{100} = 0.04.$$

To eke a little more out of this, you could note that the inequality $|A_n - \mu| \geq \epsilon$ encompasses the two symmetric cases $A_n - \mu \geq \epsilon$ and $A_n - \mu \leq -\epsilon$. (The two are symmetric in the binomial distribution, maybe not in general.) So we can divide the answer 0.04 by 2 to get 0.02 as an upper bound for the probability of getting at least 75 heads.