Homework 4 – Due July 25, 2012

Be sure to write your first and last name on your homework. Please write neatly and staple all pages together. You should show all your work!

1. Answer question 30 in the online homework. You are given the file BWurn.m. Update this file (see the question and comments in the file for more direction) and upload your code. You do not need to hand in anything on paper for this question.

2. Answer question 31 in the online homework. You are given the file exponentialPayout.m. Update this file (see the question and comments in the file for more direction) and upload your code. You do not need to hand in anything on paper for this question.

3. The goal of this problem will be to compute the expected number of *cycles* in a permutation. (If you are unfamiliar with cycles, see page 2 of the homework.)

   (a) Let Y be your random variable. Y is the number of cycles in a permutation. For $1 \leq i \leq n$, define $Y_i = 1/k$ where k is the length of the cycle containing i. Explain why $Y = Y_1 + Y_2 + \cdots + Y_n$.

   (b) Find $P(Y_i = 1/k)$. That is, what is the probability that i is in a cycle of length k? (Hint: check for $k = 1$, $k = 2$, etc.)

   (c) Find $E(Y_i)$.

   (d) Find $E(Y)$. What happens as $n \to \infty$?

4. (Section 6.2, Problem 3) You and a friend are playing roulette (see Exercises 1.1.6 and 1.1.7 for the rules). You place a 1-dollar bet on number 17 and your friend places a 1-dollar bet on black. Let X be your winnings and Y be your friends winnings. Find $E(X), E(Y), V(X)$, and $V(Y)$. Compare your findings. What do these computations tell you about the nature of your winnings if you and your friend make a sequence of bets, with you betting each time on a number and your friend betting on a color?

5. (Section 6.2, Problem 9) A die is loaded so that the probability of a face coming up is proportional to the number on that face. The die is rolled with outcome X. Find $V(X)$.

6. If X and Y are any two random variables, then the *covariance* of X and Y is defined by: $\text{Cov}(X, Y) = E((X - E(X))(Y - E(Y)))$. Covariance measures how much X and Y change together.

   (a) What is $\text{Cov}(X, X)$?

   (b) Show that if X and Y are independent, then $\text{Cov}(X, Y) = 0$.

7. (a) Let X be the number of times it takes to roll a die in order to get four 6's. What is $E(X)$ and $V(X)$?

   (b) Let Y be the number of sixes you get when you roll a die 40 times. What is $E(X)$ and $V(X)$?

   (c) Let Z be the number times you roll the die until you get your first 6. What is $E(X)$ and $V(X)$?

Cycles: A permutation σ can be decomposed into disjoint *cycles*. A cycle is constructed by taking some element of the permutation a and applying the permutation over and over until you arrive back at a (which must happen in a permutation). A k-cycle is cycle of length k. This is best understood by example. Consider the permutation

$$\sigma = 428375619 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 8 & 3 & 7 & 5 & 6 & 1 & 9 \end{pmatrix}.$$

Let $a = 1$. Then the cycle we get is (1438) because in our permutation

$$1 \mapsto 4,\ 4 \mapsto 3,\ 3 \mapsto 8, \text{ and } 8 \mapsto 1.$$

This is an example of a 4-cycle. If we choose $a = 2$, we get the cycle (2) because 2 maps to itself in the permutation (it's a fixed point). Notice that all fixed points are 1-cycles. If we take $a = 5$, then we get the 3-cycle (576). And finally if $a = 9$, we get the 1-cycle, (9).

Also note that two cycles are equal if you can cyclically rotate the numbers of one to get the other. For example,

$$(1438) = (4381) = (3814) = (8143).$$

So in this example, the cycle you get from $a = 1$ is the same cycle that you get from $a = 4, 3$, or 8.

So far, we have decomposed σ into its disjoint cycles. So we can write it as a product of cycles:

$$\sigma = (1438)(2)(576)(9).$$

We see that σ is the product of 4 cycles. There are two 1-cycles, one 3-cycle, and one 4-cycle.