Answers

Problem 1. (a) $X = \#$ of questions she answers right.

$$X_i = \begin{cases} 1 & \text{if she answers the } i^{th} \text{ question right.} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1) = \frac{65}{75} = 13/15.$$

$$E(X) = E(X_1 + \cdots + X_5) = E(X_1) + \cdots + E(X_5) = \left(\frac{13}{15}\right) \cdot 5 = \frac{13}{3}$$

$$= 4.333\cdots$$

So she should expect to pass.

(b) She passes if $P(X=4) + P(X=5)$

$$= \frac{\binom{65}{4} \cdot \binom{10}{1}}{\binom{75}{5}} + \frac{\binom{65}{5} \cdot \binom{10}{0}}{\binom{75}{5}}$$

(c) $P(\text{passes} \mid \text{at least one problem she doesn't know how to do})$

$$= \frac{P(\text{passes} + \text{one prob doesn't know})}{P(\text{one prob she doesn't know})} = \frac{\frac{\binom{65}{4} \cdot \binom{10}{1}}{\binom{75}{5}}}{1 - \frac{\binom{65}{5}}{\binom{75}{5}}}$$

(d) $P(\text{one prob doesn't know} \mid \text{passes}) = \frac{P(\text{pass} + \text{prob doesn't know})}{P(\text{passes})}$

$$= \frac{P(X=4)}{P(X=4) + P(X=5)} = \frac{\frac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5}}}{\frac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5}} + \frac{\binom{65}{5}\binom{10}{0}}{\binom{75}{5}}}$$

(e) $P(\text{Bob passes} \mid \text{April passes}) = 1 - P(\text{Bob fails} \mid \text{April passes})$

$$= 1 - \frac{P(\text{Bob fails and April passes})}{P(\text{April passes})} = 1 - \frac{P(\text{Bob fails \& Apr gets 4}) + P(\text{Bob fails \& A gets 5})}{P(\text{April passes})}$$

$$= 1 - \frac{P(\text{Bob fails \& April gets 4})}{P(\text{April passes})} = 1 - \frac{P(\text{April gets 4})P(\text{Bob fails} \mid \text{April gets 4})}{P(\text{April passes})}$$

X = April's score

Y = Bob's score.

$$P(Y>3 \mid X>3) = 1 - \frac{P(X=4)\,P(Y=3 \mid X=4)}{P(X=4 \text{ or } 5)}$$

$$P(X=4) = \frac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5}}, \quad P(X=4 \text{ or } 5) = \frac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5}} + \frac{\binom{65}{5}\binom{10}{0}}{\binom{75}{5}}.$$

$$P(Y=3 \mid X=4) = \frac{\dfrac{\binom{65}{3}\binom{10}{2}}{\binom{75}{5}}}{\dfrac{\binom{65}{3}\binom{10}{2}}{\binom{75}{5}} + \dfrac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5}} + \dfrac{\binom{65}{5}\binom{10}{0}}{\binom{75}{5}}} = \frac{\binom{65}{3}\binom{10}{2}}{\binom{65}{3}\binom{10}{2} + \binom{65}{4}\binom{10}{1} + \binom{65}{5}\binom{10}{0}}$$

Plug in to get answer

Problem session 6

II a/ (# balls in left urn)

$$\begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .25 & 0 & .75 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & .75 & 0 & .25 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = P$$

0 & 4 are absorbing states

b/

$$P = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array} & \left[\begin{array}{ccc|cc} 0 & .75 & 0 & .25 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .75 & 0 & 0 & .25 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} = \left[\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right]$$

$$I - Q = \begin{bmatrix} 1 & -.75 & 0 \\ -.5 & 1 & -.5 \\ 0 & -.75 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2.5 & 3 & 1.5 \\ 2 & 4 & 2 \\ 1.5 & 3 & 2.5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -.75 & 0 \\ -.5 & 1 & -.5 \\ 0 & -.75 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow N.(I-Q) = I \rightarrow N = (I-Q)^{-1}$

N is fundamental matrix of P

c/ We need to find $b_{3,0}$

$$B = NR = \begin{bmatrix} 2.5 & 3 & 1.5 \\ 2 & 4 & 2 \\ 1.5 & 3 & 2.5 \end{bmatrix} \cdot \begin{bmatrix} .25 & 0 \\ 0 & 0 \\ 0 & .25 \end{bmatrix} = \begin{array}{c|c} & \begin{array}{cc} 0 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} .625 & .375 \\ .5 & .5 \\ .375 & .625 \end{bmatrix} \end{array}$$

$\rightarrow b_{3,0} = .375$

d/ We need to find t_3

$$t = Nc = \begin{bmatrix} 2.5 & 3 & 1.5 \\ 2 & 4 & 2 \\ 1.5 & 3 & 2.5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 7 \end{bmatrix} \rightarrow t_3 = 7$$

Problem Session 6

#3.

(a) X = red die outcome $m(6) = 1/3$. $m(i) = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$

Y = white die outcome. $m(i) = 1/6$.

$P(\text{win}) = P(X > Y)$.

$x=6$	$y = 1,2,3,4,5$	$(\frac{1}{3}) \times (\frac{1}{6}) \times 5$
$x=5$	$y = 1\,2\,3\,4$	$\frac{2}{15} \times (\frac{1}{6}) \times 4$
$x=4$	$y = 1\,2\,3$	$\frac{2}{15} \times (\frac{1}{6}) \times 3$
$x=3$		$\frac{2}{15} \times (\frac{1}{6}) \times 2$
$x=2$		
$x=1$		$\frac{1}{15} \times (\frac{1}{6}) \times 1$

$$P(X>Y) = \frac{5}{18} + \frac{2}{15} \times \frac{1}{6} \times (1+2+3+4)$$

$$= \frac{5}{18} + \frac{10}{45} = \frac{5}{18} + \frac{2}{9} = \frac{9}{18} = \frac{1}{2}$$

$\Rightarrow$ Both dice have equal chance of winning.

(b) $E(X) = 1 \times P(X=1) + 2 \times P(X=2) + 3 \times P(X=3) + 4 \times P(X=4) + 5 \times P(X=5) + 6 \times P(X=6)$

$$= \frac{2}{3} \times \frac{1}{5} \times (1+2+3+4+5) + \frac{1}{3} \times 6 = \frac{2}{15} \times 15 + 2 = 4.$$

$$V(X) = \frac{10}{3}.$$

(c) $p = 1/3$. $q = 2/3$.

probability of rolling exactly 30 6's in 100 rolls

$$= \binom{100}{30} (1/3)^{30} (2/3)^{70}$$

(d) $p = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$, $b(100, \frac{1}{18}, 10) = \binom{100}{10} \left(\frac{1}{18}\right)^{10} \left(\frac{17}{18}\right)^{90}$

Problem 4)

a) $\dfrac{\binom{4}{1}\binom{13}{10}}{\binom{52}{10}} = P\left(\begin{array}{c}\text{all 10 cards are}\\ \text{same suit}\end{array}\right)$

b) $4!\,\dfrac{\binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{1}}{\binom{52}{10}} = P\left(4,3,2,1 \text{ suit distribution}\right)$

c) $E(X) = 10\left(\frac{13}{52}\right)^{=2.5}, \quad V(X) = np(1-p)\left(1-\frac{n-1}{N-1}\right)$ ← V(X) for hypergeometric distribution

$V(X) = 10\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(1-\frac{9}{51}\right) = \frac{30}{16}\left(\frac{42}{51}\right)$

OR, using indicator r.v.s, $X_i = \begin{cases} 1 & \text{if } i\text{th card is heart} \\ 0 & \text{o.w.}\end{cases}$

$E(X) = E(X_1 + \cdots + X_{10}) = E(X_1) + \cdots + E(X_{10}) = \frac{1}{4} + \cdots + \frac{1}{4} = \frac{10}{4}.$

$V(X) = V(X_1 + \cdots + X_{10}) = E\left((X_1 + \cdots + X_{10})^2\right) - E(X)^2 = E\left(\sum_{i=1}^{10} X_i^2 + \sum_{i=1}^{10}\sum_{j\neq i} X_iX_j\right) - E(X)^2$

$= \sum_{i=1}^{10} E(X_i^2) + \sum_{i=1}^{10}\sum_{j\neq i} E(X_iX_j) - E(X)^2 = \boxed{\frac{10}{4} + 90\cdot\frac{1}{4}\cdot\frac{12}{51} - \left(\frac{10}{4}\right)^2}$

d) 1 Ace in 5 spades AND 1 Ace in 5 hearts

$\dfrac{\binom{12}{4}}{\binom{13}{5}} \cdot \dfrac{\binom{12}{4}}{\binom{13}{5}} = P(2 \text{ Aces})$

(e)

$P(2 \text{ aces} \mid \text{at least one ace}) = \dfrac{\dfrac{\binom{4}{2}\binom{48}{8}}{\binom{52}{10}}}{1 - \dfrac{\binom{48}{10}}{\binom{52}{10}}} = \boxed{\dfrac{\binom{4}{2}\binom{48}{8}}{\binom{52}{10} - \binom{48}{10}}}$

5 (a) $p = \frac{1}{2}$

$$P\left(.45 \le \frac{S_n}{n} \le .55\right)$$

$$E\left(\frac{S_n}{n}\right) = .5 \qquad V\left(\frac{S_n}{n}\right) = \frac{pq}{n} = \frac{1}{4n}.$$

$$\cancel{V\left(\frac{S_n}{n}\right) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{1}{4n}} = \frac{1}{2\sqrt{n}}}$$

$$P\left(\frac{.45-.5}{\sqrt{\frac{1}{4n}}} \le \left(\frac{S_n}{n}\right)^* \le \frac{.55-.5}{\sqrt{1/4n}}\right)$$

$n=100$

$$= P\left(\frac{-.05}{\sqrt{1/400}} \le \left(\frac{S_n}{n}\right)^* \le \frac{.05}{\sqrt{1/400}}\right) = P\left(-.05 \cdot 20 \le \left(\frac{S_n}{n}\right)^* \le .05 \cdot 20\right)$$

$$= P\left(-1 \le \left(\frac{S_n}{n}\right)^* \le 1\right) = .68.$$

$n=10000$

$$P\left(-.05 \cdot 200 \le \left(\frac{S_n}{n}\right)^* \le .05 \cdot 200\right) = P\left(-10 \le \left(\frac{S_n}{n}\right)^* \le 10\right) \approx 1.$$

(b) $p = 1/5 \quad q = 4/5 \qquad E\left(\frac{S_n}{n}\right) = p = .2 \quad V\left(\frac{S_n}{n}\right) = \frac{.2(.8)}{n} = \frac{.16}{n}.$

$$P\left(.2 \le \frac{S_n}{n} \le .3\right)$$

$$P\left(\frac{.2-.2}{\sqrt{\frac{.16}{100}}} \le \left(\frac{S_n}{n}\right)^* \le \frac{.3-.2}{\left(\sqrt{\frac{.16}{100}}\right)}\right) = P\left(0 \le \left(\frac{S_n}{n}\right)^* \le \frac{.1}{.4/10}\right) = P\left(0 \le \left(\frac{S_n}{n}\right)^* \le 2.5\right)$$

$$= .4938.$$

(c) $E(2^X) = \sum_{i=0}^{n} 2^i \binom{n}{i} p^i q^{n-i} = \sum_{i=0}^{n} 2^i \binom{n}{i} \left(\frac{1}{2}\right)^n$

$= \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} \binom{n}{i} 2^i \underset{\substack{\uparrow \\ \text{Binomial thm}}}{=} \left(\frac{1}{2}\right)^n (1+2)^n = \left(\frac{3}{2}\right)^n.$

(d) $V(Y) = E(Y^2) - E(Y)^2 = E(2^{2X}) - E(2^X)^2$

$= E(4^X) - E(2^X)^2 = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^n 4^i - E(2^X)^2$

$= \left(\frac{1}{2}\right)^n (1+4)^n - \left(\frac{3}{2}\right)^{2n} = \left(\frac{5}{2}\right)^n - \left(\frac{9}{4}\right)^n$

$= \frac{10^n - 9^n}{4^n}.$