LECTURE 23

AUGUST 17, 2015

HAND OUT

(1) WHICH OF THE FOLLOWING TRANSITION MATRICES ARE FOR REGULAR M.C.s

(a) $\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

MULTIPLYING AGAIN GIVES THE ORIGINAL MATRIX BACK. SO FOR EVEN POWERS THE MATRIX LOOKS LIKE FOR ODD POWERS IT LOOKS LIKE THE ORIGINAL. SO THE CORRESPONDING M.C. IS NOT REGULAR.

(b) $\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$

CUBING THIS MATRIX YIELDS:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{9}{16} & \frac{1}{4} & \frac{3}{16} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}$$

SO THE CORRESPONDING M.C. IS REGULAR.

(2) (a) FOR THE M.C. FROM (1)(b), GIVEN THAT THE PROCESS STARTED IN STATE 1,

WHAT IS THE PROB IT IS IN STATE 3 AFTER 2 STEPS?

COMPUTE:

$$(1, 0, 0)\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}^2 = \left(\frac{1}{2}\ \frac{1}{3}\ \frac{1}{6}\right)$$

SO THE PROB THAT WE ARE IN STATE 3 IS $\frac{1}{6}$.

(b) FIND THE LIMITING PROB VECTOR $\vec{W}$ FOR THIS M.C.

SETUP THE FOLLOWING SYSTEM OF EQUATIONS:

$$(W_1, W_2, W_3)\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} = (W_1, W_2, W_3)$$

YIELDS:

$$\frac{W_1}{2} + \frac{3}{4}W_2 = W_1$$

$$\frac{1}{3}W_1 + W_3 = W_2$$

SETTING $W_1 = 1$ WE SOLVE FOR W_2 AND W_3 AND OBTAIN:

$(1, \frac{2}{3}, \frac{1}{3})$ DIVIDING BY $1+\frac{2}{3}+\frac{1}{3}$ YIELDS: $\boxed{\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)}$.

(3) SHOW THAT THE M.C. FOR THE FOLLOWING TRANSITION MATRIX IS ERGODIC, BUT NOT REGULAR:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

SQUARING YIELDS:

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

CUBING YIELDS

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

SO IT IS NOT REGULAR.

HOW DO WE TELL THAT IT IS ERGODIC?

LOOKING ALONG THE ROWS WE SEE THAT BY THE SECOND STEP IT IS POSSIBLE FOR ANY INITIAL STATE TO HAVE GOTTEN TO ANY OTHER. FOR INSTANCE: ON THE FIRST STEP S_1 CAN GO TO S_2. ON THE 2^{nd} STEP S_1 COULD GO TO S_1 OR S_3.

FIND THE FIXED ROW VECTOR FOR THIS CHAIN.

AGAIN COMPUTE:

$$(W_1, W_2, W_3)\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} = (W_1, W_2, W_3)$$

YIELDS: $\frac{1}{2}W_2 = W_1$, $W_1 + W_3 = W_2$, $\frac{1}{2}W_2 = W_3$

SET $W_1 = 1$ YIELDS: $W_2 = 2$, $W_3 = 1$ SO THE PROB VECTOR IS: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.