1 Sets

Sets are the most fundamental object in mathematics and the primary building block for probability. The good news is that they are something that you are already familiar with in your day-to-day life. Much of the information here can also be found in Section 1.2 of Grinstead and Snell's *Introduction to Probability.*

A *set* is a collection of things. An *element* is one of the things that lies in a set.

This intuition is all that we will need in this course - perhaps in later math or philosophy courses you will see why we actually need a slightly more precise definition of sets, but we will not worry about that here (eg. Russell's paradox). View this page as you would learning new vocabulary in a language class. We already know what these objects are, now we are giving ourselves a way to talk about them.

Notation: If we have a set S and a is an element of S, we write $a \in S$. If a is not an element of S, we write $a \notin S$.

To help us keep track of things, capital letters mean sets, i.e. A, B, S and Ω; and lowercase letters mean elements, i.e. a, b, x and ω (lowercase omega).

Examples: The set of all possible rolls of a die is $A = \{1, 2, 3, 4, 5, 6\}$. We have $1 \in A$ and $8 \notin A$. The set of all possible flips of a coin is $\Omega = \{\text{Heads}, \text{Tails}\}$.

Sets can also be infinite; $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$ is the set of all *natural numbers* and $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of *integers*.

The *empty set*, $\emptyset = \{\}$, is also a set. Lastly, the order that you write the elements in a set does not matter:

$$B = \{1, 2, 3\} = \{2, 1, 3\}.$$

Notation: Oftentimes, we may not want to list out all elements of a set, and will use the following notation to describe the set. For an example, the set

$$\Omega = \{\text{Heads}, \text{Tails}\}$$

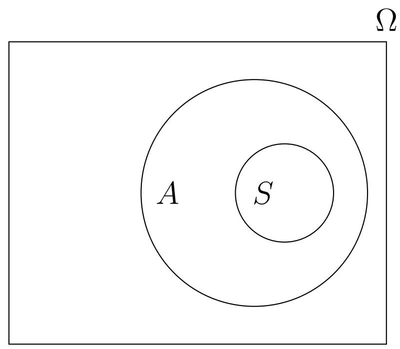
may be written as

$$\Omega = \{x \mid x \text{ is a possible outcome of a coin flip}\}.$$

We read this as "Omega is the set of elements, x, such that x is is a possible outcome of a coin flip."

2 Constructions on Sets

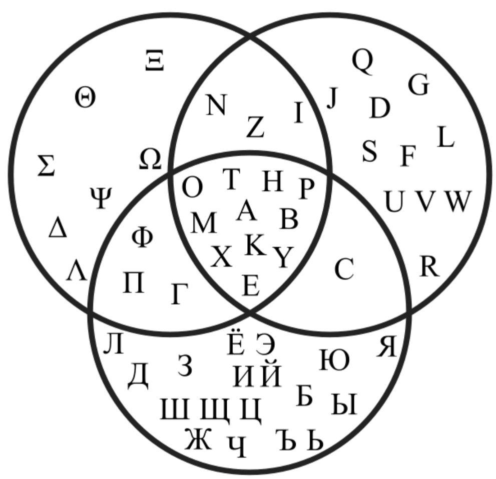
Definition: We say that S is a *subset* of A if every element of S is also an element of A. We write $S \subseteq A$, and sometimes say that "A contains S".



Example: The set $S = \{5, 7\}$ is contained in the set $A = \{1, 3, 5, 7, 18\}$ because every element that is in S is also in A.
Observation: Notice that $\emptyset$ doesn't contain any elements, and so all of its (zero) elements are in A. This tells us that $\emptyset \subseteq A$. We also have $A \subseteq A$.

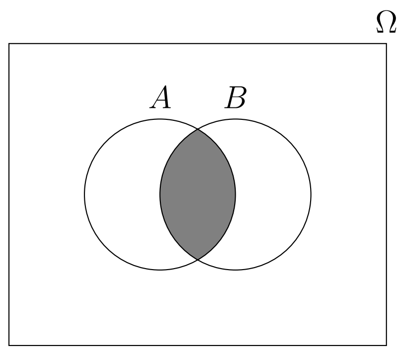
Here is an interesting Venn Diagram that I found. The sets here are Greek, Russian, and Roman letters. Call the sets $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{A}$, respectively. In particular,

$$\mathcal{A} = \{a \mid a \text{ is a capital Roman letter}\}.$$



Example: The set $\{T, H, P\} \subseteq \mathcal{A}$. The set $\{\Sigma, Z, \Omega, E\} \subseteq \mathcal{G}$. However, $\{\Theta, C\}$ is not a subset of $\mathcal{G}$ because $C \notin \mathcal{G}$. We write $\{\Theta, C\} \nsubseteq \mathcal{G}$ to say that C is not a subset of $\mathcal{G}$. The set $\{O, T, H, P, M, A, B, X, K, Y, E\}$ is interesting because it is a subset of $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{A}$. In this example, this set is the intersection of the three sets.
Exercise: List all of the subsets of $A = \{1, 2, 3\}$. Rembember $\emptyset$ and A itself are subsets of A. How many are there?
Definition: The *intersection* of two sets is the set of all of the elements that are in both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

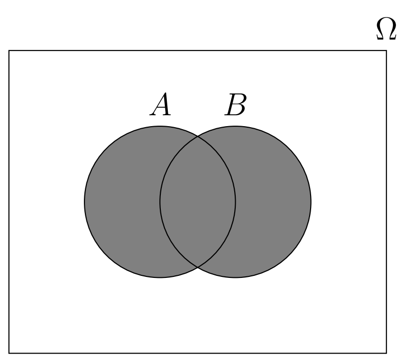


Example: The intersection of $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 4, 8\}$ is

$$A \cap B = \{1, 2, 4\}.$$

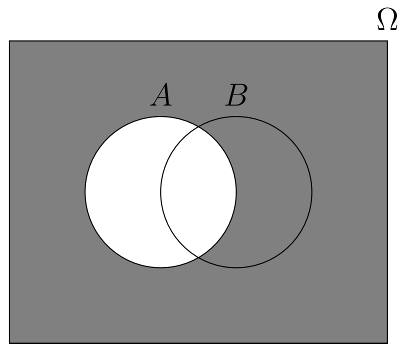
Mnemonic: The symbol $\cap$ makes $A \cap B$ look similar to the word "A∩D".
Definition: The *union* of two sets is the set of all elements that are in A or B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$



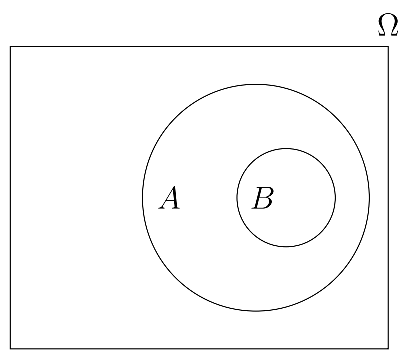
Defintion: Let Ω be a larger universal set, say $\Omega = \{1, 2, 3, 4, 5, 6\}$, the set of possible rolls of a die. We have the subset $A = \{1, 3, 5\}$ of odd rolls. Then $A^C = \{2, 4, 6\}$, the set of alll elements of Ω that are not in A, is called the *complement* of A in Ω. We write A^C (as long as the set Ω is understood).

$$A^C = \{x \in \Omega \mid x \notin A\}$$



Examples: The complement of the set $B = \{1, 2\}$ in $\Omega = \{1, 2, 3, 4, 5, 6\}$ is $B^C = \{3, 4, 5, 6\}$. The complement of $\{1, 3, 5, 7, 9, \ldots\}$ in $\mathbb{N}$ is $\{2, 4, 6, 8, 10, \ldots\}$.
Problem: Show the following: If $B \subseteq A$, then $B \cap A = B$.

Solution: Since B is a subset of A, then the region for B must be contained completely in the region A. This is because every element of B must also be an element of A. This gives the drawing for the Venn Diagram below.



First shade the set A with veritical lines, and the set B with horizontal lines (do this). We are interested in $B \cap A$. These are the elements that are in both B and A, which means that they will be those elements in the regions that have been shaded in both directions. Looking at the picture, the region that has been shaded in both directions is exactly the set B. We can now say $B \cap A = B$.

Assignment: Please turn in on a separate sheet of paper. Please explain your reasoning for both questions in a few complete sentences. The presentation of your understanding is important.

1. Using Venn Diagrams (as in the problem on the previous page), show that
$$(A \cup B)^c = A^c \cap B^c.$$
2. What is another way of representing $(A \cap B)^c$? Explain why you know this is true using Venn Diagrams.