Math 20 – Problem Set 1 (due July 6)

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

2. How many ways can 5 distinct trophies may be awarded to 30 students if no student may receive more than one trophy?

3. How many ways are there to distribute 5 identical marbles among 8 children such that no child receives more than one? What if the children may receive more than one?

4. How many different ways can the letters of the following words be arranged?

   (a) AUSTIN

   (b) DALLAS

   (c) SAN ANTONIO (don't worry about the space)

5. From a 52-card playing deck, how many 5-card hands contain at least one card of every suit?

6. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents.

   (a) How many committees are possible?

   (b) How many committees are possible if each *party* must elect a leader among their representatives on the committee?

7. Prove the hockey-stick identity for binomial coefficients: for all positive integers n and k such that $k \leq n$,
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$
by induction on n. *(Hint: This is induction on n, not k. For the base case, you only have to consider one value of k. In the inductive step, let n AND k be arbitrary.)* Why is this called the hockey-stick identity?

8. *(Optional)* Let $A, B \subseteq \Omega$. Prove that $\#A + \#B = \#(A \cap B) + \#(A \cup B)$ as follows:

(a) Let $x \in A \cap B^c$. How much does the point x contribute to each side of the above equality? (eg. on the left side, how many of the sets A and B contain x, and on the right side, how many of the sets $A \cap B$ and $A \cup B$ contain x?)

(b) Repeat (a) with an element $y \in A^c \cap B$.

(c) Repeat (a) with an element $z \in A \cap B$.

(d) Repeat (a) with an element $w \in (A \cup B)^c$.

(e) Show that the union of the four sets in parts (a)-(d) contains all of Ω.