Math 20 – Problem Set 1 Solutions

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. *Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?*

   Every handshake takes place between two people (order does not matter), so there are $\binom{20}{2} = 190$ total handshakes.

2. *How many ways can 5 distinct trophies may be awarded to 30 students if no student may receive more than one trophy?*

   There are 30 ways to award the first trophy (pick any student to give it to), 29 ways to award the second (pick any child except the one who received the first), etc., for a total of ${}_{30}P_5 = \frac{30!}{25!}$ ways to award the trophies. Note that order matters here because the trophies are distinct; if they were identical, the answer would be $\binom{30}{5}$.

3. How many ways are there to distribute 5 identical marbles among 8 children such that no child receives more than one? What if the children may receive more than one?

   If each child may receive only one marble, we just choose 5 children to receive one marble each, so the answer is $\binom{8}{5}$.

   If the children may receive more than one, this is a "stars and bars" type problem, where we have to choose an unordered subset (the children to whom we give each marble) with repetitions allowed. There are $n = 8$ children and $k = 5$ marbles, so the answer is $\binom{n+k-1}{k} = \binom{12}{5}$

4. *How many different ways can the letters of the following words be arranged?*

   (a) AUSTIN
   (b) DALLAS
   (c) SAN ANTONIO (don't worry about the space)

   (a) 6! (all letters are distinct)

(b) $\binom{6}{1,2,2,1} = \binom{6}{1} \cdot \binom{5}{2} \cdot \binom{3}{2} \cdot \binom{1}{1} = \frac{6!}{1!\cdot 2!\cdot 2!\cdot 1!}$

(c) $\binom{10}{1,2,3,1,2,1} = \binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{1}\binom{3}{2}\binom{1}{1} = \frac{10!}{1!\cdot 2!\cdot 3!\cdot 1!\cdot 2!\cdot 1!}$

5. *From a 52-card playing deck, how many 5-card hands contain at least one card of every suit?*

   If a hand contains each of the 4 suits and exactly 5 cards, there is one suit with two cards, and the other suits must each have one. Therefore the answer is $\binom{4}{1}\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1} = 4 \cdot 13^4 \cdot 6$, as we choose the suit with two cards, then two cards from that suit, then one card from each of the remaining suits.

6. *A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents.*

   (a) *How many committees are possible?*

   (b) *How many committees are possible if each* party *must elect a leader among their representatives on the committee?*

   (a) Positions on a committee are treated as identical, so the order in which we choose the members does not matter. Therefore we have $\binom{5}{2}\binom{6}{2}\binom{4}{3}$ ways to choose the committee.

   (b) There are a couple of different ways to do this problem. Using the answer above, we can now choose, among the two elected Republicans, one of them to be the leader, and similarly with the Democrats and Republicans. This gives us $\binom{5}{2}\binom{6}{2}\binom{4}{3} \cdot 2 \cdot 2 \cdot 3$ ways to form the committee.

   Another way to proceed would be to choose the leader from each party first, then fill the remaining seats. If we count this way, there are $5 \cdot 4$ ways to choose the Republicans, $6 \cdot 5$ ways to choose the Democrats, and $4 \cdot \binom{3}{2}$ ways to choose the Independents for a total of $5 \cdot 4 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ total ways to choose the committee. You can check that this gives the same answer as above (7200).

7. *Prove the hockey-stick identity for binomial coefficients: for all positive integers n and k such that $k \leq n$,*

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

   by induction on n. (Hint: This is induction on n, not k. For the base case, you only have to consider one value of k. In the inductive step, let n AND k be arbitrary.) *Why is this called the hockey-stick identity?*

Let S_n be the statement "for all integers k such that $1 \leq k \leq n$, $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$. We proceed by induction on n.

For the base case $n = 1$, we need only consider the case where $k = 1$. We have

$$\sum_{i=1}^{1} \binom{i}{1} = \binom{1}{1} = 1 = \binom{2}{2}$$

as desired.

Let n be arbitrary and suppose S_n is true. We need to show S_{n+1}: for every k such that $1 \leq k \leq n+1$, $\sum_{i=1}^{n+1} \binom{i}{k} = \binom{n+2}{k+1}$. Let k be any positive integer between 1 and n. Then we have

$$\begin{aligned} \sum_{i=k}^{n+1} \binom{i}{k} &= \sum_{i=k}^{n} \binom{i}{k} + \binom{n+1}{k} \\ &= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{(by the induction hypothesis)} \\ &= \binom{n+2}{k+1} \quad \text{(by Pascal's Relation)} \end{aligned}$$

as desired, so we are done by induction.

(Minor point: We actually haven't proven it for the case $k = n + 1$, but in this case, both sides of the equality are equal to 1.)

This is called the hockey-stick identity because of the shape the terms in the equality form in Pascal's Triangle – draw it for yourself to see!