Math 20 – Problem Set 2 (due July 11)

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. *Suppose two cards are drawn from a standard 52-card playing deck without replacement. Determine which of the following pairs of events are independent. Justify your answers using the independence equation, not just intuition!*

   (a) A : *The first card is a spade.*
   B : *The second card is a club.*

   (b) C : *The first card is black (a spade or a club).*
   D : *The second card is a club.*

   (c) E : *The first card is a face (Jack, Queen, or King).*
   F : *The second card is a face.*

   (a) $P(A) = \frac{1}{4} = P(B)$, but $P(B|A) = \frac{12}{51} \neq P(B)$, so these events are dependent.

   (b) $P(C) = \frac{1}{2}$, $P(D) = \frac{1}{4}$, and $P(B|A) = \frac{12}{51} \neq P(B)$, so these events are dependent.

   (c) $P(E) = \frac{12}{52} = P(F)$, but $P(F|E) = \frac{11}{52} \neq P(F)$, so these events are dependent.

2. *In Yahtzee, five fair 6-sided dice are rolled and points are awarded for different combinations. A* large straight *is a result in which the five dice take the distinct values* $\{1, 2, 3, 4, 5\}$ *or* $\{2, 3, 4, 5, 6\}$. *What is the probability of rolling a large straight on a single roll of the five dice?*

   Notice that in this problem we have to consider choosing the values of the dice *in order*, because unordered sequences of die rolls (as opposed to ordered sequences) are not equally likely (for example, you're more likely to roll 4 1's and a 2 than to roll 5 1's). The number of ways we can roll $\{1, 2, 3, 4, 5\}$ on the 5 dice (in some order) is 5!, and this is the same as the number of ways to roll $\{2, 3, 4, 5, 6\}$ on the 5 dice (in some order). Since the total number of possible results is 6^5, we get an answer of

   $$\frac{2 \cdot 5!}{6^5}.$$

3. *In chess, rooks can move any distance either vertically or horizontally, but not diagonally, and if one piece can move to the square of another piece, it can "attack" that piece. Suppose 8 rooks are randomly placed on a (8-by-8) chessboard with at most one rook per square. What is the probability that none of the rooks can attack another?*

*(*Hint: How many ways can the rooks be placed on the board? What must the board look like if no rook can attack another?*)*

If two rooks are placed in such a way that they can't attack each other, they must occupy different rows and columns. Thus eight non-attacking rooks must each occupy different rows and columns, so we can think of them as forming a permutation of the numbers one through eight: we pick a row for the leftmost rook, then we have one fewer choice for the rook in the second column, etc. There are 8! such rooks and $\binom{64}{8}$ ways to place them for a probability of

$$\frac{8!}{\binom{64}{8}}$$

that the placement is non-attacking.

4. *Two archers, Jack and Diane, each shoot a single arrow at a target. Suppose Jack hits the target 60% of the time and Diane hits the target 90% of the time.*

   (a) *What is the probability that both archers hit the target? What assumption are you making?*

   (b) *Making the same assumption as above, what is the probability that exactly one of the archers hits the target?*

   (c) *Under the same assumption, if exactly one of the two arrows hits the target, what is the probability it was Jack who fired this arrow?*

   (d) *Instead of the assumption you made for parts (a)-(c), suppose instead that Diane fires first, and if she hits the target, Jack's probability of hitting the target drops to 30%, since Jack might hit Diane's arrow. Now what is the probability Jack hits the target after Diane takes her shot?*

   (a) We assume their shots are independent. Then $P(J \cap D) = P(J)P(D) = 0.6 \cdot 0.9 = 0.54$.

   (b)

$$\begin{aligned}
P(\text{exactly one hits}) &= P((J \cap D^c) \cup (J^c \cap D)) \\
&= P(J)(1 - P(D)) + (1 - P(J))P(D) \\
&= 0.6 \cdot 0.1 + 0.4 \cdot 0.9 \\
&= 0.06 + 0.36 \\
&= 0.42
\end{aligned}$$

(c) We use Bayes' Theorem:

$$\begin{aligned}
P(J|\text{exactly one hits}) &= P(J \cap \text{exactly one hits})/P(\text{exactly one hits}) \\
&= P(J \cap D^c)/0.42 \\
&= 0.6 \cdot 0.1/0.42 \\
&= \frac{1}{7}
\end{aligned}$$

(d) Now we're not assuming the shots are independent, but instead that $P(J|D) = 0.3$ and $P(J|D^c) = 0.6$. Thus

$$\begin{aligned}
P(J) &= P(J|D)P(D) + P(J|D^c)P(D^c) \\
&= 0.3 \cdot 0.9 + 0.6 \cdot 0.1 \\
&= 0.27 + 0.06 \\
&= 0.33.
\end{aligned}$$

5. *Show that for any two events A and B with $P(A) > 0$ and $P(B) > 0$, if $P(A|B) > P(A)$, then also $P(B|A) > P(B)$. (*Hint: How are these four quantities related?*)*

Proof: Begin with the observation that $P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$, so also $P(A|B)/P(A) = P(B|A)/P(B)$. If $P(A|B) > P(A)$, then $P(A|B)/P(A) > 1$ and so necessarily $P(B|A)/P(B) > 1$, but then $P(B|A) > P(B)$.

6. (a) *Box A contains 4 red balls and 4 blue balls, and Box B contains 2 red balls and 4 blue balls. One of the two boxes is selected at random with equal probability, and then a ball is drawn randomly from the selected box. If the ball is red, what is the probability that the selected box was Box A?*

(b) *Suppose now that two balls are drawn from the same box selected in part (a), without replacement. If both balls are red, what is the probability that the selected box was Box A?*

(a) Let A be the event that Box A was selected and let C be the event that a red ball was drawn. Given in the problem we have $P(A) = \frac{1}{2}$, $P(C|A) = \frac{1}{2}$, and $P(C|A^c) = \frac{1}{3}$. By Bayes' Theorem,

$$\begin{aligned}
P(A|C) &= \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|A^c)P(A^c)} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}} \\
&= \frac{\frac{1}{4}}{\frac{5}{12}} \\
&= \frac{3}{5}.
\end{aligned}$$

(b) Here we essentially repeat the calculation from part (a), but instead of C being the event that a red ball was drawn, C is the event that TWO red balls were drawn without replacement. Thus $P(C|A) = \frac{1}{2} \cdot \frac{3}{7} = \frac{3}{14}$ and $P(C|A^c) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$. Then we can compute

$$P(A|C) = \frac{\frac{1}{2}\frac{3}{14}}{\frac{1}{2}\frac{3}{14} + \frac{1}{2}\frac{1}{15}} = \frac{\frac{3}{14}}{\frac{59}{210}} = \frac{45}{59}$$