Math 20 – Problem Set 4 (due July 25)

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. Suppose you flip a penny and a dime, and their outcomes are independent. Let X be the result of flipping the penny where we assign the value of Heads to be 1 and the value of Tails to be 0, and let Y be the result of flipping the dime where we assign the value of Heads to be 10 and the value of Tails to be 0. Compute the following:

   (a) $E(X + Y)$

   (b) $E(XY)$

   (c) $\text{Var}(X + Y)$

   (d) $\text{Var}(XY)$.

2. If A is any event and X is a discrete random variable with sample space Ω, the *conditional expectation of X given A* is defined by

$$E(X|A) = \sum_{x \in \Omega} xP(X = x|A)$$

   where $P(X = x|A)$ is the probability that $X = x$ given that the event A occurs. Roll two fair six-sided dice and let X be the sum of their faces. Compute the following:

   (a) $E(X|A)$ where A is the event that the first die lands on 1 or 2.

   (b) $E(X|A^c)$ where A is as above.

   (c) $E(X|A)P(A) + E(X|A^c)P(A^c)$ where A is as above.

3. A random variable X has the following distribution:

$$P(X = 0) = \frac{1}{3}, \quad P(X = 1) = \frac{1}{3}, \quad P(X = 2) = \frac{1}{6}, \quad P(X = 3) = \frac{1}{6}.$$

   Compute $E(X), \text{Var}(X)$, and $\text{sd}(X)$.

4. Let X be a discrete random variable that takes only positive integer values. In this case, our definition for expected values tells us that

$$E(X) = \sum_{k=1}^{\infty} kP(X = k).$$

Prove the following alternate formula:

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

which will be useful for us later in the class. This doesn't need to be a rigorous proof; in fact, a picture may help!

5. Use the formula from (4) to give an alternate proof of the fact that if X is a geometric random variable with parameter p, then

$$E(X) = \frac{1}{p}.$$

(Hint: You'll need an expression for $P(X \geq k)$. If we think of the geometric distribution as describing the number of coin flips it takes to land heads, where each time the result is heads with probability p, what has to happen on the first $k - 1$ coin flips for this to be the case?)

6. (a) The average number of homes sold per day by a specific real estate company is two. What is the probability that the company will sell exactly three homes tomorrow?

   (b) What assumptions did you make in your answer to part (a)? Why might this be incorrect?