Math 20 – Problem Set 4 (due August 1)

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. *Suppose you flip a penny and a dime, and their outcomes are independent. Let X be the result of flipping the penny where we assign the value of Heads to be 1 and the value of Tails to be 0, and let Y be the result of flipping the dime where we assign the value of Heads to be 10 and the value of Tails to be 0. Compute the following:*

   (a) $E(X + Y)$

   (b) $E(XY)$

   (c) $\mathrm{Var}(X + Y)$

   (d) $\mathrm{Var}(XY)$.

   Note that $E(X) = 0.5$ and $E(Y) = 5$. Also, $\mathrm{Var}(X) = \frac{1}{2}(0 - 0.5)^2 + \frac{1}{2}(1 - 0.5)^2 = 0.25$, and $\mathrm{Var}(Y) = \mathrm{Var}(10X) = 100 \cdot \mathrm{Var}(X) = 25$.

   (a) By linearity, $E(X + Y) = E(X) + E(Y) = 5.5$.

   (b) Since X and Y are independent, $E(XY) = E(X)E(Y) = 2.5$.

   (c) Again, since X and Y are independent, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 25.25$.

   (d) For this problem, we should compute the probability distribution for XY. Notice that if either coin comes up tails, $XY = 0$; this happens with probability 0.75. If both coins land heads, then $XY = 10$; this happens with probability 0.25. Then we compute

$$\begin{aligned}
\mathrm{Var}(XY) &= E((XY - 2.5)^2) \\
&= 0.75 \cdot 2.5^2 + 0.25 \cdot 7.5^2 \\
&= \frac{75}{16} + \frac{225}{16} \\
&= \frac{75}{4}.
\end{aligned}$$

2. *If A is any event and X is a discrete random variable with sample space Ω, the* conditional expectation of X given A *is defined by*

$$E(X|A) = \sum_{x \in \Omega} x P(X = x|A)$$

   where $P(X = x|A)$ is the probability that $X = x$ given that the event A occurs. Roll two fair six-sided dice and let X be the sum of their faces. Compute the following:

(a) *$E(X|A)$ where A is the event that the first die lands on 1 or 2.*

(b) *$E(X|A^c)$ where A is as above.*

(c) *$E(X|A)P(A) + E(X|A^c)P(A^c)$ where A is as above.*

The event A is the set of ordered pairs of dice such that the first is either a one or a two, so we have $A = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(2,6)\}$. Summing these dice, there is one way to get 2 and 8, and two ways each for 3, 4, 5, 6, and 7.

(a) $E(X|A) = \frac{1}{12}(2+8) + \frac{2}{12}(3+4+5+6+7) = 5$.
Alternate (slick) proof: $E(X|A) = E(\text{first die}|A) + E(\text{second die}|A) = 1.5 + 3.5 = 5$.

(b) A similar computation using A^c gives $E(X|A^c) = 8$.

(c) $E(X|A)P(A) + E(X|A^c)P(A^c) = 5 \cdot \frac{1}{3} + 8 \cdot \frac{2}{3} = \frac{21}{3} = 7$. In general, $E(X|A)P(A) + E(X|A^c)P(A^c) = E(X)$.

3. *A random variable X has the following distribution:*

$$P(X=0) = \frac{1}{3},\ \ P(X=1) = \frac{1}{3},\ \ P(X=2) = \frac{1}{6},\ \ P(X=3) = \frac{1}{6}.$$

Compute $E(X), \mathrm{Var}(X)$, and $\mathrm{sd}(X)$.

$E(X) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 = \frac{7}{6}$.
$\mathrm{Var}(X) = \frac{1}{3}(0-\frac{7}{6})^2 + \frac{1}{3}(1-\frac{7}{6})^2 + \frac{1}{6}(2-\frac{7}{6})^2 + \frac{1}{6}(3-\frac{7}{6})^2 = \frac{41}{36}$
$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = \frac{\sqrt{41}}{6}$.

4. *Let X be a discrete random variable that takes only positive integer values. In this case, our definition for expected values tells us that*

$$E(X) = \sum_{k=1}^{\infty} kP(X=k).$$

Prove the following alternate formula:

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

which will be useful for us later in the class. This doesn't need to be a rigorous proof; in fact, a picture may help!

Consider the following triangle:

$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + \cdots$
$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + \cdots$
$P(X \geq 3) = P(X = 3) + P(X = 4) + \cdots$
$P(X \geq 4) = P(X = 4) + \cdots$
$\vdots$

Summing the diagonals of this triangle, we get exactly $\sum_{k=1}^{\infty} kP(X = k)$, which is our definition of $E(X)$. Summing the rows instead, we get $\sum_{k=1}^{\infty} P(X \geq k)$. Since each term is counted once in each expression (as long as these sums converge), they must be the same!

5. *Use the formula from (4) to give an alternate proof of the fact that if X is a geometric random variable with parameter p, then*

$$E(X) = \frac{1}{p}.$$

(Hint: You'll need an expression for $P(X \geq k)$. If we think of the geometric distribution as describing the number of coin flips it takes to land heads, where each time the result is heads with probability p, what has to happen on the first $k-1$ coin flips for this to be the case?)

To say that it takes at least k coin flips for one to land heads is exactly the same as saying that the first $k-1$ coin flips all landed tails! The probability that this happens is exactly $(1-p)^{k-1}$. Then by 4,

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} P(X \geq k) \\ &= \sum_{k=1}^{\infty} (1-p)^{k-1} \\ &= \frac{1}{1-(1-p)} \\ &= \frac{1}{p}. \end{aligned}$$

6. (a) *The average number of homes sold per day by a specific real estate company is two. What is the probability that the company will sell exactly three homes tomorrow?*
   (b) *What assumptions did you make in your answer to part (a)? Why might this be incorrect?*

(a) We model this situation with a Poisson distribution with rate $\lambda = 2$ houses per day. Then $P(X = 3) = \frac{e^{-2}2^3}{3!} = \frac{4}{3e^{-2}}$.

(b) To use the Poisson distribution, we must assume that occurrences (home sales) occur at a consistent rate, and independently. This may not be quite accurate, as, for example, rates of home sales depend on the time of year, and may occur together (ie. if a builder sells several new homes in the same new neighborhood).