Math 20 – Problem Set 5 (due August 1)

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. *Let X be a random variable with range $[-1, 1]$ and let $f(x)$ be its density of X. Find μ_X and σ_X^2, if, for $|x| \le 1$:*

   (a) $f(x) = \frac{3}{4}(1 - x^2)$
   (b) $f(x) = \frac{x+1}{2}$
   (c) $f(x) = \frac{3}{8}(x + 1)^2$

   (This is #2 a,c,d from Grinstead and Snell, page 277.)

   (a) $E(X) = \int_{-1}^{1} x\frac{3}{4}(1 - x^2)dx = \int_{-1}^{1}(\frac{3}{4}x - \frac{3}{4}x^3)dx = (\frac{3}{2}x^2 - \frac{3}{16}x^4)|_{-1}^{1} = (\frac{3}{2} - \frac{3}{16}) - (\frac{3}{2} - \frac{3}{16}) = 0$.
   $\text{Var}(X) = \int_{-1}^{1}(x^2 - 0)\frac{3}{4}(1 - x^2)dx = \int_{-1}^{1} \frac{3}{4}(x^2 - x^4)dx = (\frac{1}{4}x^3 - \frac{3}{20}x^5)|_{-1}^{1} = (\frac{1}{4} - \frac{3}{20}) - (-\frac{3}{4} + \frac{3}{20}) = \frac{1}{5}$.
   (b) $E(X) = \frac{1}{3}$; $\text{Var}(X) = \frac{2}{9}$.
   (c) $E(X) = \frac{1}{2}$; $\text{Var}(X) = \frac{3}{20}$.

2. *Let X be a random variable defined on the interval $[0, \pi]$ whose density function is $f(x) = \frac{\sin(x)}{2}$. Compute the CDF $F(x)$. What is $P(\frac{\pi}{6} \le X \le \frac{\pi}{2})$?*

   $F(x) = \int_0^x f(t)dt = \int_0^x \frac{\sin(t)}{2}dt = \frac{-\cos(t)}{2}|_0^x = \frac{1}{2} - \frac{\cos(x)}{2}$

   $P(\frac{\pi}{6} \le x \le \frac{\pi}{2}) = F(\frac{\pi}{2}) - F(\frac{\pi}{6}) = \frac{\cos(\pi/6) - \cos(\pi/2)}{2} = \frac{\sqrt{3}}{4}$.

3. *On an average 8-hour school day, 960 people walk into Kemeny Hall. Assume, though this is certainly not the case, that this happens randomly at a constant rate over the 8 hours. What is the probability that exactly 8 people walk into Kemeny Hall within a 10-minute interval during the school day? What is the probability that exactly 48 people walk into Kemeny Hall within an hour?*

   We model this with a Poisson distribution. The rate at which students enter the building is 960 per 8 hours, or 960/8/6 = 20 per 10 minutes. The probability that 8 people walk in during a 10-minute interval is $\frac{e^{-20}20^8}{8!}$.

For the second part, considering an hour interval, the rate is 120, so the probability that 48 people enter is $\frac{e^{-120}120^{48}}{48!}$.

4. *The* half-life *of an isotope is the amount of time it takes for the probability of one isotope to decay into another to be 50%. The time it takes for a Carbon-14 isotope to decay into a Nitrogen-14 isotope is given by an exponential distribution with expected value estimated at 8267 years. Find its half-life to the nearest year. Use a calculator to simplify exponentials and logarithms. (Note: Decay of radioactive particles is probably the most approprate process to model with the exponential distribution.)*

   We want to find the value x for which $P(X \leq x) = \frac{1}{2}$. But we know $P(X \leq x) = 1 - e^{\frac{-1}{8267}x}$ as this is the CDF of the exponential distribution. Then we have

$$\begin{aligned} \frac{1}{2} &= e^{\frac{-1}{8267}x} \\ 2 &= e^{\frac{1}{8267}x} \\ \ln(2) &= \frac{1}{8267}x \\ x &= \ln(2) \cdot 8267 \approx 5730 \text{ years.} \end{aligned}$$

5. *Suppose the height of an adult male is given by a normal distribution with expected value 70 inches and standard deviation 4 inches.*

   (a) *Shaquille O'Neal is 83 inches tall. What proportion of adult males are taller than Shaq? Use a standard normal distribution table.*

   (b) *Darren Sproles is 66 inches tall. What proportion of adult males are shorter than Darren? Do you need a normal distribution table to answer this question?*

   (a) $Z = \frac{X-\mu}{\sigma} = \frac{83-70}{4} = 3.25; P(X \geq 83) = 1 - \Phi(3.25) \approx 0.0006.$

   (b) $X = \frac{X-\mu}{\sigma} = \frac{66-70}{4} = -1; P(X \leq -1) = \Phi(-1) \approx 0.1587.$

6. Prove that if X is a continuous random variable with range $[x_1, x_2]$ and finite expected value μ, then,

$$\text{Var}(X) = E(X^2) - \mu^2.$$

   (Note: This is true for *all* discrete and continuous RVs with finite mean and variance, and the proof is almost exactly the same!)

$$
\begin{aligned}
\text{Var}(X) &= \int_{x_1}^{x_2} (x-\mu)^2 f(x)dx \\
&= \int_{x_1}^{x_2} (x^2 - 2\mu x + \mu^2) f(x)dx \\
&= \int_{x_1}^{x_2} x^2 f(x)dx + 2\mu \int_{x_1}^{x_2} x f(x)dx + \mu^2 \int_{x_1}^{x_2} f(x)dx \\
&= E(X^2) + 2\mu \cdot \mu - \mu^2 \cdot 1 \\
&= E(X^2) - \mu^2
\end{aligned}
$$

An easier way: $\text{Var}(X) = E((X-\mu)^2) = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(x) + \mu^2 = E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$. Notice that this proof doesn't require X to be continuous – it works more generally.