Math 20 – Problem Set 6 (due August 15)

This problem set is due at the *beginning* of class. This is just the problem list; please work out these problems on a different sheet of paper. Please write neatly, staple the pages together, and explain your work where appropriate. You do not need to simplify binomial coefficients $\binom{n}{k}$ for both which $k > 3$ and $n - k > 3$, or exponentials n^k where $n + k > 8$.

1. *Midterm 2, #16.*

   (a) *Prove that for a random variable X with expected value $E(X) = \lambda$,*

   $$\mathrm{Var}(X) = E(X(X-1)) + \lambda - \lambda^2.$$

   Do not use the equality $\mathrm{Var}(X) = E(X^2) - E(X)^2$ unless you prove it.

   (b) *Use part (a) to prove that the standard deviation of the Poisson distribution with parameter λ is $\sqrt{\lambda}$. You may use without proof that its expected value is λ. (This was 16(b) from your exam.)*

   (a)

   $$\begin{aligned}
   \mathrm{Var}(X) &= E((X-\lambda)^2) \\
   &= E(X^2 - 2\lambda X + \lambda^2) \\
   &= E(X^2 - X + X) - 2\lambda E(X) + \lambda^2 \\
   &= E(X(X-1)) + E(X) - 2\lambda^2 + \lambda^2 \\
   &= E(X(X-1)) + \lambda - \lambda^2.
   \end{aligned}$$

   (b) Let X be a Poisson random variable with parameter λ. Then

   $$\begin{aligned}
   E(X(X-1)) &= \sum_{k=0}^{\infty} k(k-1)P(X=k) \\
   &= \sum_{k=2}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} \\
   &= \lambda^2 \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^{k-2}}{(k-2)!} \\
   &= \lambda^2 \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{k!} \\
   &= \lambda^2.
   \end{aligned}$$

   Then using a), $\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$, so $\mathrm{sd}(X) = \sqrt{\lambda}$.

2. *Suppose X has PDF $f(x) = \frac{1}{2}$ on $[3,5]$ and 0 otherwise and Y has PDF $g(y) = \frac{1}{2}(y-3)$ on $[3,5]$ and 0 otherwise. Find the PDF of $X+Y$.*

   $(f*g)(z) = \int_{-\infty}^{\infty} f(x)g(z-x)dx = \int_{-\infty}^{\infty} \frac{1}{4}(z-x-3)dx$.

   The domain of X and Y tell us that $3 \leq x \leq 5$ and $3 \leq z-x \leq 5$, and thus $3 \leq x$, $x \leq 5$, $x-3 \leq z$, and $z \leq x-5$. If $6 \leq z \leq 8$, the relevant conditions are $3 \leq x \leq z-3$, so we have

$$(f*g)(x) = \int_3^{z-3} \frac{1}{4}(z-x-3)dx = \frac{1}{8}(z-6)^2.$$

   If $8 \leq z \leq 10$, then the relevant conditions are $z-5 \leq x \leq 5$, so we have

$$(f*g)(x) = \int_{z-5}^{5} \frac{1}{4}(z-x-3)dx = -\frac{1}{8}(z-6)(z-10).$$

   In summary,

$$(f*g)(z) = \begin{cases} 0 & z < 6 \text{ or } z > 10 \\ \frac{1}{8}(z-6)^2 & 6 \leq z \leq 8 \\ -\frac{1}{8}(z-6)(z-10) & 8 \leq z \leq 10 \end{cases}$$

3. *Let $S_n = X_1 + X_2 + \cdots + X_n$ is a sum of n independent exponentially distributed random variables with rate λ. Prove by induction (remember this?) that for all $n \geq 2$, S_n has PDF*

$$f_{S_n}(z) = \frac{\lambda e^{-\lambda z}(\lambda z)^{n-1}}{(n-1)!}.$$

   (Hint: $S_n = S_{n-1} + X_n$.)

   For the base case $n = 2$, we proved in class that the PDF of a sum of two exponential random variables is $\lambda^2 z e^{-\lambda z}$ on $[0, \infty)$. When $n = 2$, we have

$$f_{S_2}(z) = \frac{\lambda e^{-\lambda z}(\lambda z)}{1!} = \lambda^2 z e^{-\lambda z}.$$

   Now assume that $f_{S_{n-1}}(z) = \frac{\lambda e^{-\lambda z}(\lambda z)^{n-2}}{(n-2)!}$; we want to show that S_n has the PDF above.

By induction we have

$$\begin{aligned}
f_{S_n}(z) = (f_{S_{n-1}} * f_{X_n})(z) &= \int_{-\infty}^{\infty} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-2}}{(n-2)!} \cdot \lambda e^{-\lambda(z-x)} dx \\
&= \frac{\lambda^n}{(n-2)!} \int_0^z e^{-\lambda(x+z-x)} x^{n-2} dx \\
&= \frac{\lambda^n e^{-\lambda z}}{(n-2)!} \int_0^z x^{n-2} dx \\
&= \frac{\lambda \cdot \lambda^{n-1} e^{-\lambda z}}{(n-2)!} \cdot \frac{z^{n-1}}{n-1} \\
&= \frac{\lambda e^{-\lambda z}(\lambda z)^{n-1}}{(n-1)!}
\end{aligned}$$

4. *A particle takes a biased random walk on the integer number line as follows: Starting at 0, it moves one to the left with probability 0.2 and one to the right with probability 0.8 at each step. After 2500 such steps, find the expected value of its position and the standard deviation of its position, and estimate (using the Central Limit Theorem) the probability that the particle lands within 10 steps of its expected position.*

   For each $i \in \{1, 2, \ldots, 2500\}$, let X_i be a random variable taking the value -1 if the particle steps to the left and 1 if the particle steps to the right. Then for each i, $E(X_i) = -1 \cdot 0.2 + 1 \cdot 0.8 = 0.6$ and $\text{Var}(X_i) = 0.2 \cdot (-1 - 0.6)^2 + 0.8 \cdot (1 - 0.6)^2 = 0.64$.

   Let S_{2500} be the position of the particle after 2500 steps, so $S_{2500} = \sum_{i=1}^{2500} X_i$. Then $E(S_{2500}) = 2500 \cdot 0.6 = 1500$ and $\text{sd}(S_{2500}) = \sqrt{2500 \cdot 0.64} = 40$.

   The Central Limit Theorem tells us that $\frac{S_n - 1500}{40}$ is approximately normally distributed, so

   $$P(1490 \le S_n \le 1510) \approx P(\frac{-10.5}{40} \le Z \le \frac{10.5}{40}) = \Phi(0.2625) - \Phi(-0.2625) = 0.2052.$$

5. *Estimate the probability that the sum of 100 independently chosen random number drawn uniformly from the interval $[0, 1]$ is greater than 51. Then estimate the probability that the sum of 10000 independently chosen random numbers drawn uniformly from the interval $[0, 1]$ is greater than 5100. Comment on the disparity between these probabilities.*

   The expected value and variance of a single randomly chosen number on the unit interval are $\frac{1}{2}$ and $\frac{1}{12}$, respectively, so we have

   $$P(S_{100} > 51) \approx P(Z > \frac{(51 - 50) \cdot \sqrt{12}}{10}) = P(Z > 0.3464\ldots) = 1 - 0.6331 = 0.3669.$$

$$P(S_{10000} > 5100) \approx P(Z > \frac{(5100-5000)\cdot\sqrt{12}}{100}) = P(Z > 3.464\ldots) = 1-0.9997 = 0.0003.$$

Perhaps surprisingly, the probability that S_n takes a value larger than 1% of its expected value decreases drastically as $n \to \infty$. However, we know this to be the case – this is guaranteed by the Law of Large Numbers!