Math 20: Probability

Final

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`https://github.com/fudab/Math-20`

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Your name: ____

Instructions

- Please **print your name** on the first page of your answer sheet.
- This exam has **eight questions**. Some questions have multiple parts; please be mindful of this and write your answers in the **order** of the questions.
- Present your work **neatly and clearly**. LaTeX, Microsoft Word or other text editing tool is welcomed. Justify your answers **completely**. Unless explicitly told otherwise, you will not receive full credit for insufficiently justified answers. Please box your answers, when appropriate.
- It is fine to leave your answer in a form such as $\phi(a)$ or $\text{NA}(a, b)$. However, if an expression can be easily simplified (such as $2+3$, $e^{\ln(3)}$ or $\binom{6}{2}$), please simplify it.
- You need to complete the exam **independently**. Use of Canvas, slides, homework, quizzes, recordings, textbooks as well as code is permitted.
- Sign below (or transcribe this passage on your answer sheet) to indicate your adherence to the honor code:

I, ____, have neither given nor received unauthorized help on this exam, and I have conducted myself within the guidelines of the Academic Honor Principle. Moreover, I will not discuss the content of this exam with anyone until authorized to do so.

Problem 1: True or False

10 = 2 + 2 + 2 + 2 + 2 pts

(a) ______ For any two numerically-valued random variable X and Y, $V(X+Y) > V(X) + V(Y)$.

(b) ______ For any two numerically-valued random variable X and Y, if X and Y are independent, $V(X+Y) = V(X-Y)$.

(c) ______ Let X be a discrete random variable with finite range $\{0,1\}$. Then $\mu_1 = \mu_2 = \cdots = \mu_k = \cdots$, where μ_k is the kth moment.

(d) ______ Let X be a nonnegative discrete random variable with ordinary generating function $h_X(z)$. Suppose that $Y = bX$. Then $h_Y(z) = h_X(bz)$.

(e) ______ Let P be the transition matrix of a Markov chain. The sum of the entries in a single row of the matrix P^n can be less than 1.

Problem 2: Computation

13 = 3 + 5 + 5 pts

Let X and Y be two independent random variables.

(a) Suppose that X and Y are both discrete with common distribution

$$\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

Find the probability $P(X+Y=2)$.

(b) Suppose that X and Y are both continuous with density functions

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{y}{2}, & 0 < y < 2 \\ 0. & \text{otherwise} \end{cases}$$

- Consider the sum $Z = X + Y$. Find its density function $f_Z(z)$.

  Reference: $\int x e^{cx} dx = e^{cx} \frac{cx-1}{c^2}$.

- Consider the difference $Z = X - Y$. Find its density function $f_Z(z)$.

Problem 3: Proof

10 = 5 + 5 pts

(a) Let S_n be the number of success in n Bernoulli trials with probability p for success on each trial. Show that the estimate

$$P(|\frac{S_n}{n} - p| \geq \epsilon) \leq \frac{1}{4n\epsilon^2}.$$

(b) Let X be a non-negative continuous random variable with density function $f(x)$ and moments μ_n. For any positive real number a, prove that

$$P(0 \leq X \leq a) \geq 1 - \frac{\mu_n}{a^n}.$$

Problem 4: Manipulation

17 = 2 + 4 + 4 + 4 + 3 pts

Let X be a discrete random variable with range $\{1, 2, 3, \cdots\}$, distribution function p, and ordinary generating function $h(z) = \frac{z}{2-z}$.

(a) Find the probability $\mathrm{P}(X = 1)$.

(b) Find the expected value $E(X)$ and the variance $V(X)$.

(c) Find the distribution function p. What can you say about X?

Let $X_1, X_2, \cdots, X_n$ be an identical independent trials process. Assume that X_i's all have the same distribution as X. Consider the sum $S_n = X_1 + X_2 + \cdots + X_n$.

(d) Find the moment generating function $g_{S_n}(t)$ and the ordinary generating function $h_{S_n}(z)$ of S_n.

Hint: A fast pass would be generalizing an identity we have seen in class. If you figure out which identity it is, you can directly use it.

(e) Find the probability $P(S_n = n + 1)$.

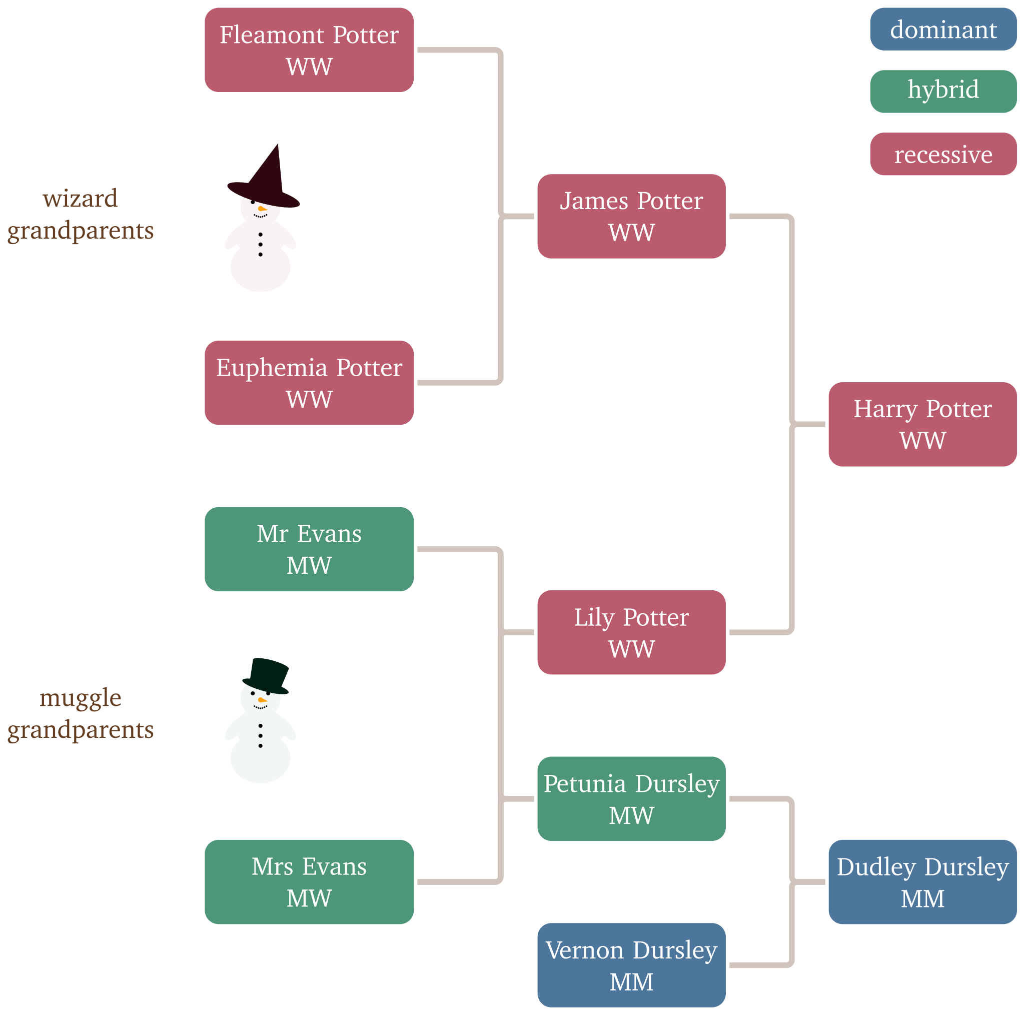
Note: There are multiple ways to solve this part. Choose a pain-free one.

Problem 5: Genetics of Wizardry

12 = 8 + 4 pts

The simplest type of inheritance of wizardry occurs when it is governed by a pair of genes, each of which may be of two types: M (muggle) and W (wizard).

An individual may have a MM combination or MW (which is genetically the same as WM) or WW. The MM and MW types are indistinguishable, and we say that the M (muggle) gene dominates the W (wizard) gene. An individual is called dominant if he or she has MM genes, recessive if he or she has WW and hybrid with a MW mixture.



Harry Potter family tree

There is a muggle couple live in Hanover and both the husband and the wife are hybrid with a MW mixture.

(a) Find the expected value and the variance for the number of muggle children in the family until there is a wizard child.

Hint: You can make use of a discrete distribution we learnt in class.

(b) Find the probability distribution for the number of muggle children in the family until there is a wizard child or until there are three children, whichever comes first.

Problem 6: Wandcraft

12 = 3 + 3 + 3 + 3 pts

Garrick Ollivander is a British wizard who is the proprietor of Ollivanders in Diagon Alley. He is widely considered the best wandmaker in the world.

Wands contain magical cores, possibly magically inserted once the wand has been carved. Mr Ollivander opts to only use phoenix feathers, unicorn hairs, and dragon heartstrings, which may be the most powerful and best of magical cores, as Ollivander's wands have been praised by many witches and wizards from around the world.

phoenix feathers

unicorn hairs

dragon heartstrings

Ollivanders serves 100 customers per year. On the average phoenix feathers are used in 20 percent of the wands, unicorn hairs in 30 percent of the wands, and dragon heartstrings in 50 percent of them.

Note: Please present your answers in the form of $\phi(a)$ or $\mathrm{NA}(a, b)$ for (a), (b) and the second part of (c).

(a) Estimate the probability that Mr Ollivander sells exactly 50 wands with dragon heartstrings in a year.

(b) Estimate the probability that Mr Ollivander sells at least 30 wands with unicorn hairs in a year.

(c) Estimate the probability that Mr Ollivander sells at least 16 and at most 24 wands with phoenix feathers in a year.

- Use Law of Large Numbers (Chebyshev's inequality). What can you say about the result?

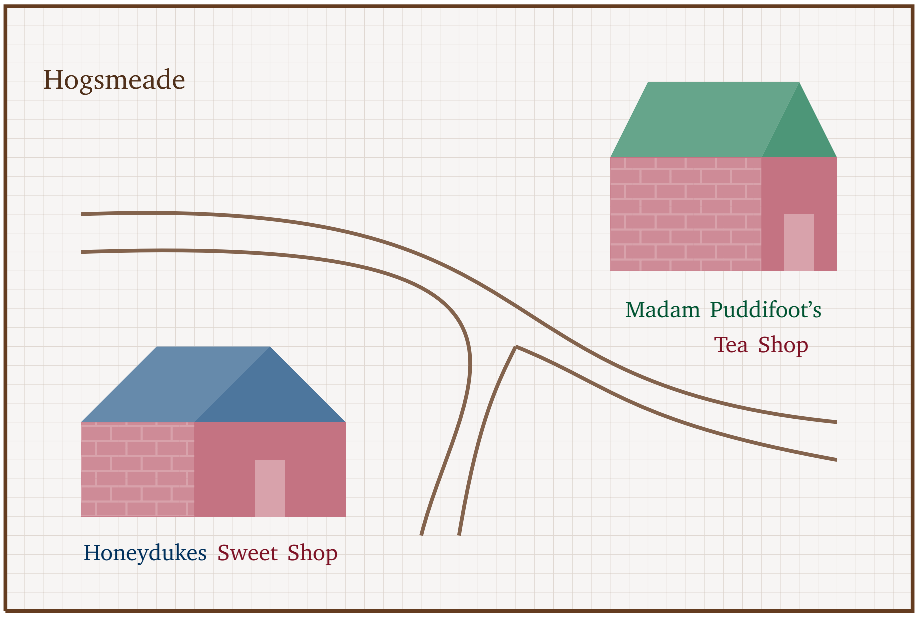
- Use Central Limit Theorem.

Problem 7: Hogsmeade Weekend Trips

13 = 3 + 4 + 3 + 3 pts

Hogsmeade weekend trips are special trips that the students at Hogwarts School of Witchcraft and Wizardry could take on certain weekends to Hogsmeade village.

On a regular Sunday, the number of students that arrive at the village in unit time has a Poisson distribution with parameter $\lambda = 10$. After arriving, a student chooses to visit Honeydukes Sweet Shop with probability $\frac{3}{5}$ and Madam Puddifoot's Tea Shop with probability $\frac{2}{5}$.



Hogsmeade map

(a) Let N be the number of students that arrive at the village in unit time. Find the ordinary generating function $h_N(z)$ for N.

(b) Let X_i be the indicator random variable for the ith student that arrives at the village in unit time. To be more specific,

$$X_i = \begin{cases} 1, & \text{if the } i\text{th student visit Madam Puddifoot's Tea Shop} \\ 0. & \text{if the } i\text{th student visit Honeydukes Sweet Shop} \end{cases}$$

Find the ordinary generating function $h_X(z)$ for X_i.

(c) Let S be the number of students that visit Madam Puddifoot's Tea Shop in unit time.

Theorem

Suppose $X_1, X_2, \cdots$ is a sequence of independent random variables with common distribution function. Let N be a random variable, independent of the X_i's with ordinary generating function $h_N(z)$. If $S = X_1 + X_2 + \cdots + X_N$, then the ordinary generating function of S is

$$h_S(z) = h_N(h_X(z)).$$

- Use the above theorem to find the ordinary generating function $h_S(z)$ for S.

- Explain why S also has a Poisson distribution with parameter $\lambda = 4$.

Problem 8: Genetics of Wizardry (Continued)

13 = 4 + 3 + 3 + 3 pts

Recall the rules of probability in Problem 5. Consider now a process of continued matings. It starts with an muggle man (MM or MW) or a wizard (WW) who marries a witch (WW). Assume that they have at least one son. A son is chosen at random and he again marries a witch (WW) and the process repeated through a number of generations.

The genetic type of the chosen son in successive generations can be represented by a Markov chain. The set of states is $S = \{s_1, s_2, s_3\} = \{\text{dominant}, \text{hybrid}, \text{recessive}\}$.

States

(a) Find the transition matrix P.

(b) Assume that the grandfather is a hybrid muggle man. Find the probability that the grandson is also a hybrid muggle man.

(c) Assume that the grandfather is a dominant muggle man with probability $\frac{1}{4}$, a hybrid muggle man with probability $\frac{1}{2}$, and a wizard with probability $\frac{1}{4}$. Find the probability that the grandson is a hybrid muggle man.

(d) The Noble and Most Ancient House of Black is one of the largest, oldest, and wealthiest wizard families in Britain. The Black family traces its origin back to the Middle Ages when a dominant muggle man married a witch. For generations, the men in the family only marry witches. What do you think would the male-line family tree look like after centuries of matings?

'Some people are worth melting for.' Olaf (*Frozen*)

You are the reason that I enjoy teaching during the summer. I wish you a successful conclusion of the term, a restful break, and the very best for all the new ventures.