

Math 20: Probability

Midterm 1

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<https://github.com/fudab/Math-20>

July 20, 2020

Your name: _____



Instructions

- Please **print your name** on the first page of your answer sheet.
- This exam has **nine questions**. Some questions have multiple parts; please be mindful of this and write your answers in the **order** of the questions.
- Present your work **neatly and clearly**. ~~TEX~~ LaTeX, Microsoft Word or other text editing tool is welcomed. Justify your answers **completely**. Unless explicitly told otherwise, you will not receive full credit for insufficiently justified answers. Please box your answers, when appropriate.
- It is fine to leave your answer in a form such as $\binom{80}{20}$. However, if an expression can be easily simplified (such as $2 + 3$, $e^{\ln(3)}$ or $\binom{6}{2}$), please simplify it.
- You need to complete the exam **independently**. Use of Canvas, slides, notes, recordings, textbooks as well as code is permitted.
- Sign below (or transcribe this passage on your answer sheet) to indicate your adherence to the honor code:

I, _____, have neither given nor received unauthorized help on this exam, and I have conducted myself within the guidelines of the Academic Honor Principle. Moreover, I will not discuss the content of this exam with anyone until authorized to do so.



Problem 1: True or False

10 pts

- (a) _____ For any two events A and B , $P(A \cap B) \geq 1 - P(A \cup B)$.
- (b) _____ For a random variable T following an exponential distribution $f(t) = \lambda e^{-\lambda t}$, $P(T = \frac{\ln(2)}{\lambda}) = \frac{\lambda}{2}$.
- (c) _____ For any event E , $P(E | E^c) = 1$.
- (d) _____ For any two independent events A and B , $P(A | B) = P(A | B^c)$.
- (e) _____ As $n \rightarrow \infty$, both the ratio and the difference of $n!$ and $n^n e^{-n} \sqrt{2\pi n}$ (Stirling's Formula) approach 0.



Problem 2: Computation

10 pts

(a) Compute the following:

- $\binom{20}{17}$

- $B(6, 0.2, 2)$

(b) Find the integers n and r such that the following equation is true:

$$\binom{20}{9} + 2\binom{20}{10} + \binom{20}{11} = \binom{n}{r}.$$



Problem 3: Proof

15 pts

(a) Prove that for any positive integer $n \geq 1$,

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} = \binom{2n}{1} + \binom{2n}{3} + \binom{2n}{5} + \cdots + \binom{2n}{2n-1}.$$

This identity is another proof for Question 5 in Quiz 4.

(b) Prove that for any three events A, B, C , all having positive probability, and with the property that $P(A \cap B) > 0$,

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B).$$

(c) Suppose that A and B are events such that $P(A|B) = P(B|A)$. Also $P(A \cup B) = \frac{1}{5}$ and $P(A \cap B) > 0$. Prove that $P(B) > \frac{1}{10}$.



Problem 4: Manipulation

10 pts

Let X, Y be random numbers chosen independently from the interval $[0, 1]$ with uniform distribution.

- (a) Let $Z = X^2 + Y^2$. For $Z \leq 1$, find the cumulative distribution function and the density function of Z .

- (b) Given that $0 \leq |X - Y| \leq \frac{1}{2}$, find the probability that $Z \leq \frac{1}{4}$.



Problem 5: National Committee of Senators

10 pts

In the United States, a state is a constituent political entity, of which there are currently 50. Bound together in a political union, each state is represented in the Senate (irrespective of population size) by two senators.



Two senators from the same state

We consider the events that a national committee of 50 senators are chosen at random. Please find the probability, respectively, for:

(a) New Hampshire is represented.

(b) All states are represented.



Problem 6: Star Trek: Long and Prosper

10 pts

Duck Musk's company **SpaceD** has sent out 10 starships to search for extraterrestrial life among a target set of n exoplanets.

Due to a '**Swan**' program error, however, each of these 10 starships is drifting to one randomly chosen destination out of these n target exoplanets.



Starship officers

- (a) Suppose $n = 10$. Derive the probability that these starships each will have landed in different exoplanets.
- (b) Our universe is vast. Please derive an approximation for the smallest number for n such that these starships each will have landed in different destinations with probability greater than 0.99.

Hint from science officer **Spock**:

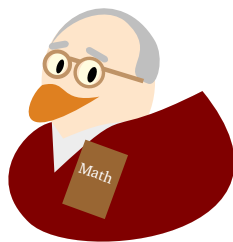
Stirling's formula $n! \sim n^n e^{-n} \sqrt{2\pi n}$ and Taylor expansion $\log(1+x) \sim x - \frac{x^2}{2}$.



Problem 7: Role Playing Game (RPG)

10 pts

The **Duckmouth** is a 2020 role-playing game developed and published by **Math 20: Probability** and is based on the book **Introductkion to Probability**. Players control protagonist **Duck D Random**, a mathematician who is looking for his missing daughter **Duckota C Random**.



Duck D Random



Duckota C Random

The character has four core attributes: Intelligence (I), Wisdom (W), Charisma (C) and Strength (S). Before the game starts, attribute scores are determined randomly by distributing character points. The values for these four attributes satisfying the three conditions:

- the sum is fixed to 10 points,
- the value of any attribute is no less than 1 point,
- the value of any attribute is no greater than 4 points.

An example of the attribute sequence is given below.



I: 4 points



W: 3 points



C: 2 points



S: 1 points

In the initialization stage, how many possible ways are there in total to customize the character?






Problem 8: Role Playing Game (RPG) continued

10 pts

In the RPG game **Duckmouth**, you will also be assigned another character as your assistant **Quack Random** before the adventure (like **Holmes** and **Watson**).

There are 100 candidates in the backend, vary in rareness and power. Among these characters, 75 are Class R (rare), 20 are Class SR (super rare), and only 5 are Class SSR (specially super rare).

During the storyline, you need to send your assistant to undertake different missions. For R, SR and SSR characters, their chances of accomplishing a mission is shown in the table.

R	SR	SSR
		
25%	50%	100%

Now the server randomly drops a character as your assistant. In the first chapter of the game, your assistant has completed 4 missions in a row. What is the probability that he or she will accomplish the next mission?



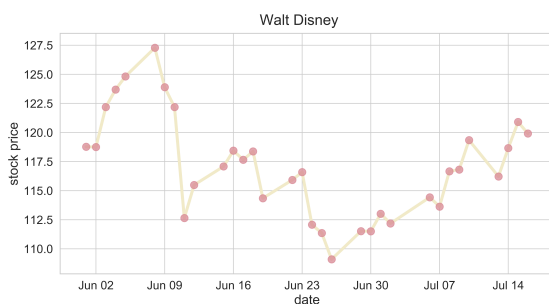
Problem 9: A Random Walk Down Wall Street

15 pts

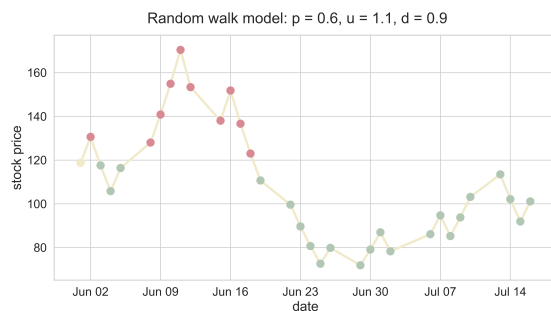
The Hedge Fund firm **Renaissance Ducknologies** is developing a stock price forecasting system.

The chief technology officer **Leonardo duck Vinci** would like to modify the original random walk model

$$S(t + 1) = \begin{cases} uS(t), & \text{with probability } p \\ dS(t). & \text{with probability } 1 - p \end{cases}$$



Actual stock price



Simulation result

Academics have not conclusively proved whether the stock market truly operates like a random walk or is based on predictable trends. There have been many published studies that support or undermine both sides of the issue.

As a consultant skilled in probability theory, you are asked to provide constructive suggestions to improve the model and hence the stock price may more closely resemble the simulations. Please present your (no more than one-page) proposal.

The code of the original model is available in the [Github repository](#) for your reference.

```
random_walk_sp(n = 30, p = 0.6, c = 100, u = 1.1, d = 0.9)
path_rw_sp(company_index = 'DIS', date_initial = datetime.date(int(2020),int(6),int(1)), p = 0.6, u = 1.1, d = 0.9,
           fsize = (12, 6), fs = 20, index = 1)
```

