Math 20: Probability

Midterm 1 Solution

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https://github.com/fudab/Math-20

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Problem 1: True or False

10 pts

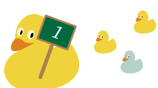
(a) False For any two events A and B, $P(A \cap B) \geq 1 - P(A \cup B)$.

(b) False For a random variable T following an exponential distribution $f(t) = \lambda e^{-\lambda t}$, $P(T = \frac{\ln(2)}{\lambda}) = \frac{\lambda}{2}$.

(c) False For any event E, $P(E \mid E^c) = 1$.

(d) True For any two independent events A and B, $P(A \mid B) = P(A \mid B^c)$.

(e) False As $n \to \infty$, both the ratio and the difference of $n!$ and $n^n e^{-n} \sqrt{2\pi n}$ (**Stirling's Formula**) approach 0.



Problem 2: Computation

10 pts

(a) Compute the following:

- $\binom{20}{17}$

3 pts

$$\binom{20}{17} = \binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140.$$

- $B(6, 0.2, 2)$

3 pts

$$B(6, 0.2, 2) = \binom{6}{2}(\frac{1}{5})^2(\frac{4}{5})^4 = \frac{768}{3125} = 0.24576.$$

(b) Find the integers n and r such that the following equation is true:

$$\binom{20}{9} + 2\binom{20}{10} + \binom{20}{11} = \binom{n}{r}.$$

4 pts

Given that

$$\binom{20}{9} + 2\binom{20}{10} + \binom{20}{11} = \binom{21}{10} + \binom{21}{11} = \binom{22}{11},$$

we get $n = 22$ and $r = 11$.

Problem 3: Proof

15 pts

(a) Prove that for any positive integer $n \geq 1$,

$$\binom{2n}{0}+\binom{2n}{2}+\binom{2n}{4}+\cdots\binom{2n}{2n}=\binom{2n}{1}+\binom{2n}{3}+\binom{2n}{5}+\cdots\binom{2n}{2n-1}.$$

This identity is another proof for Question 5 in Quiz 4.

5 pts

Consider the **Binomial Theorem** $(a+b)^n = \sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}$.

Let $a=-1$, $b=1$ and use $2n$ instead of n. We have

$$0=\binom{2n}{0}-\binom{2n}{1}+\binom{2n}{2}-\binom{2n}{3}+\cdots-\binom{2n}{2n-1}+\binom{2n}{2n},$$

which further leads to the original identity.

(b) Prove that for any three events A, B, C, all having positive probability, and with the property that $P(A\cap B)>0$,

$$P(A\cap B\cap C)=P(A)P(B\,|\,A)P(C\,|\,A\cap B).$$

5 pts

Apply the formula for conditional probability $P(F\,|\,E)=\frac{F\cap E}{P(E)}$.

$$P(A\cap B\cap C)=P(A\cap B)P(C\,|\,A\cap B)=P(A)P(B\,|\,A)P(C\,|\,A\cap B).$$

(c) Suppose that A and B are events such that $P(A|B)=P(B\,|\,A)$. Also $P(A\cup B)=\frac{1}{5}$ and $P(A\cap B)>0$. Prove that $P(B)>\frac{1}{10}$.

5 pts

Notice that $P(A|B)=\frac{P(A\cap B)}{P(B)}$ and $P(B\,|\,A)=\frac{P(B\cap A)}{P(A)}$.

If $P(A|B)=P(B\,|\,A)$, we immediately get $P(A)=P(B)$.

Since

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=2P(B)-P(A\cap B)=\frac{1}{5}$$

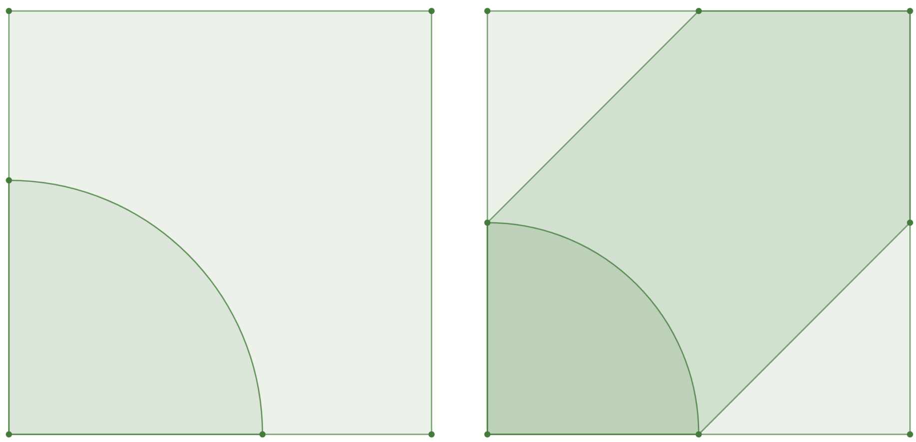
and $P(A\cap B)>0$, it follows that

$$P(B)=\frac{1}{2}[P(A\cup B)+P(A\cap B)]>\frac{1}{10}.$$

Problem 4: Manipulation

10 pts

Let X, Y be random numbers chosen independently from the interval $[0, 1]$ with uniform distribution.



(a) Let $Z = X^2 + Y^2$. For $Z \leq 1$, find the cumulative distribution function and the density function of Z.

5 pts

$$F_Z(z) = P(Z \leq z) = P(X^2 + Y^2 \leq z) = \frac{\pi z}{4}.$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{\pi}{4}.$$

(b) Given that $0 \leq |X - Y| \leq \frac{1}{2}$, find the probability that $Z \leq \frac{1}{4}$.

5 pts

$$P(Z \leq \frac{1}{4} | 0 \leq |X - Y| \leq \frac{1}{2}) = \frac{\frac{\pi}{16}}{\frac{3}{4}} = \frac{\pi}{12}.$$

Problem 5: National Committee of Senators

10 pts

In the United States, a state is a constituent political entity, of which there are currently 50. Bound together in a political union, each state is represented in the Senate (irrespective of population size) by two senators.

Two senators from the same state

We consider the events that a national committee of 50 senators are chosen at random. Please find the probability, respectively, for:

(a) New Hampshire is represented.

5 pts

$$p = 1 - \frac{\binom{98}{50}}{\binom{100}{50}} = 1 - \frac{98!50!50!}{100!50!48!} = 1 - \frac{50 \times 49}{100 \times 99} = \frac{149}{198}.$$

(b) All states are represented.

5 pts

$$p = \frac{2^{50}}{\binom{100}{50}}.$$

Problem 6: Star Trek: Long and Prosper

10 pts

Duck Musk's company **SpaceD** has sent out 10 starships to search for extraterrestrial life among a target set of n exoplanets. Due to a **'Swan'** program error, however, each of these 10 starships is drifting to one randomly chosen destination out of these n target exoplanets.

Starship officers

(a) Suppose $n = 10$. Derive the probability that these starships each will have landed in different exoplanets.

5 pts

$$p = \frac{10!}{10^{10}}.$$

(b) Our universe is vast. Please derive an approximation for the smallest number for n such that these starships each will have landed in different destinations with probability greater than 0.99.

Hint from science officer **Spock**: Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ and Taylor expansion $\log(1+x) \sim x - \frac{x^2}{2}$.

5 pts

$$p = \frac{n!}{n^{10}(n-10)!} \approx \frac{n^n e^{-n}\sqrt{2\pi n}}{n^{10}(n-10)^{n-10}e^{-n+10}\sqrt{2\pi(n-10)}}$$

$$= (\frac{n}{n-10})^{n-10+\frac{1}{2}}e^{-10} = (1+\frac{10}{n-10})^{n-10+\frac{1}{2}}e^{-10}.$$

$$\ln(p) = (n-10+\frac{1}{2})\ln(1+\frac{10}{n-10}) - 10 \approx (n-10+\frac{1}{2})[\frac{10}{n-10} - \frac{50}{(n-10)^2}] - 10$$

$$\approx (10 - \frac{45}{n-10}) - 10 = -\frac{45}{n-10} > \ln(0.99).$$

That is, $n > -\frac{45}{\ln(0.99)} + 10$. The smallest n is 4488.

Problem 7: Role Playing Game (RPG)

10 pts

The **Duckmouth** is a 2020 role-playing game developed and published by **Math 20: Probability** and is based on the book **Introducktion to Probability**. Players control protagonist **Duck D Random**, a mathematician who is looking for his missing daughter **Duckota C Random**.

Duck D Random

Duckota C Random

The character has four core attributes: Intelligence (I), Wisdom (W), Charisma (C) and Strength (S). Before the game starts, attribute scores are determined randomly by distributing character points. The values for these four attributes satisfying the three conditions:

- the sum is fixed to 10 points,
- the value of any attribute is no less than 1 point,
- the value of any attribute is no greater than 4 points.

An example of the attribute sequence is given below.

I: 4 points

W: 3 points

C: 2 points

S: 1 points

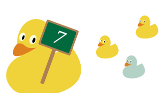
In the initialization stage, how many possible ways are there in total to customize the character?

10 pts

$$10 = 1+1+4+4 = 1+2+3+4 = 1+3+3+3 = 2+2+2+4 = 2+2+3+3.$$

The number of ways is

$$\binom{4}{2} + 4! + \binom{4}{1} + \binom{4}{1} + \binom{4}{2} = 44.$$



Problem 8: Role Playing Game (RPG) continued

10 pts

In the RPG game **Duckmouth**, you will also be assigned another character as your assistant **Quack Random** before the adventure (like **Holmes** and **Watson**).There are 100 candidates in the backend, vary in rareness and power. Among these characters, 75 are Class R (rare), 20 are Class SR (super rare), and only 5 are Class SSR (specially super rare). During the storyline, you need to send your assistant to undertake different missions. For R, SR and SSR characters, their chances of accomplishing a mission is shown in the table.

R	SR	SSR
		
25%	50%	100%

Now the server randomly drops a character as your assistant. In the first chapter of the game, your assistant has completed 4 missions in a row. What is the probability that he or she will accomplish the next mission?

10 pts

Let E and F be the events that the assistant completed 4 missions in a row and that he or she will complete the next mission. We have

$$P(E) = P(E|R)P(R) + P(E|SR)P(SR) + P(E|SSR)P(SSR) = (\frac{1}{4})^4 \frac{75}{100} + (\frac{1}{2})^4 \frac{20}{100} + (1)^4 \frac{5}{100}.$$

And

$$P(R|E) = \frac{P(R)P(E|R)}{P(E)},\ P(SR|E) = \frac{P(SR)P(E|SR)}{P(E)},\ P(SSR|E) = \frac{P(SSR)P(E|SSR)}{P(E)}.$$

After some calculation, we get $P(R|E) = \frac{15}{335} = \frac{3}{67}$, $P(SR|E) = \frac{64}{335}$, $P(SSR|E) = \frac{256}{335}$. Therefore,

$$\begin{aligned} P(F) &= P(F|R)P(R) + P(F|SR)P(SR) + P(F|SSR)P(SSR) \\ &= \frac{1}{4} \times \frac{15}{335} + \frac{1}{2} \times \frac{64}{335} + 1 \times \frac{256}{335} = \frac{1167}{1340}. \end{aligned}$$

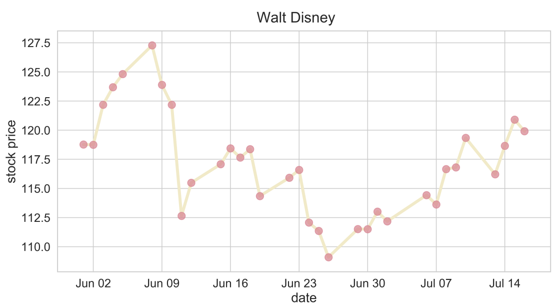
Problem 9: A Random Walk Down Wall Street

15 pts

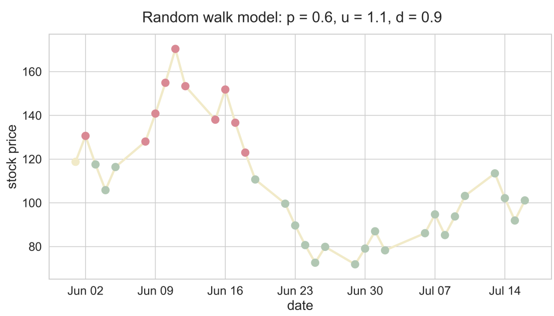
The Hedge Fund firm **Renaissance Ducknologies** is developing a stock price forecasting system.

The chief technology officer **Leonardo duck Vinci** would like to modify the original random walk model

$$S(t+1) = \begin{cases} uS(t), & \text{with probability } p \\ dS(t). & \text{with probability } 1-p \end{cases}$$



Actual stock price



Simulation result

Academics have not conclusively proved whether the stock market truly operates like a random walk or is based on predictable trends. There have been many published studies that support or undermine both sides of the issue.

As a consultant skilled in probability theory, you are asked to provide constructive suggestions to improve the model and hence the stock price may more closely resemble the simulations. Please present your proposal in details.

The code of the original model is available in the Github repository for your reference.

```
random_walk_sp(n = 30, p = 0.6, c = 100, u = 1.1, d = 0.9)
path_rw_sp(company_index = 'DIS', date_initial = datetime.date(int(2020),int(6),int(1)), p = 0.6, u = 1.1, d = 0.9,
          fsize = (12, 6), fs = 20, index = 1)
```