Math 20: Probability

Midterm 2

Instructor: Xingru Chen

https://github.com/fudab/Math-20

August 10, 2020

Your name: ______________________________

Instructions

- Please **print your name** on the first page of your answer sheet.
- This exam has **eight questions**. Some questions have multiple parts; please be mindful of this and write your answers in the **order** of the questions.
- Present your work **neatly and clearly**. LaTeX, Microsoft Word or other text editing tool is welcomed. Justify your answers **completely**. Unless explicitly told otherwise, you will not receive full credit for insufficiently justified answers. Please box your answers, when appropriate.
- It is fine to leave your answer in a form such as $\binom{80}{20}$. However, if an expression can be easily simplified (such as $2+3$, $e^{\ln(3)}$ or $\binom{6}{2}$), please simplify it.
- You need to complete the exam **independently**. Use of Canvas, slides, notes, recordings, textbooks as well as code is permitted.
- Sign below (or transcribe this passage on your answer sheet) to indicate your adherence to the honor code:

I, ______________________________, have neither given nor received unauthorized help on this exam, and I have conducted myself within the guidelines of the Academic Honor Principle. Moreover, I will not discuss the content of this exam with anyone until authorized to do so.

Problem 1: True or False

10 = 2 + 2 + 2 + 2 + 2 pts

(a) ______ For any two numerically-valued random variable X and Y, if X and Y are independent, X^2 and Y^2 are also independent.

(b) ______ Let X be a random variable which can take on values $\{0, 1, 2, 3, ...\}$ (all non-negative integers). X can be uniformly distributed.

Hint: The sum of countably many zeros is still zero.

(c) ______ For a numerically-valued random variable X, if $E(X^2) = E^2(X)$, X can take two different values.

(d) ______ For any two numerically-valued random variable X and Y, if $\text{COV}(X, Y) = 0$, X and Y are independent.

(e) ______ For any numerically-valued random variable X, its expected value and variance cannot be negative.

Problem 2: Computation

13 = 3 + 5 + 5 pts

Let X and Y be two **independent** random variables.

(a) Assume that X is the outcome of a single Bernoulli trial where $m(x) = \begin{cases} p, & X = 1 \\ 1-p, & X = 0 \end{cases}$.

Find $E(X^{2020})$.

(b) Assume that X is Poisson distributed with parameter λ. Find $E(X^3)$.

(c) Assume that X and Y are exponentially distributed with parameter λ_1 and λ_2. Find $E((X+Y)^2)$.

Problem 3: Proof

17 = 2 + 5 + 5 + 5 pts

Let X be the outcome of a chance experiment with $E(X) = \mu$ and $V(X) = \sigma^2$. When μ and σ are unknown, the statistician often estimates them by repeating the experiment n times with outcome $x_1, x_2, \cdots, x_n$, estimating μ by the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

and σ^2 by the sample variance

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Show that, for the sample mean $\bar{x}$ and sample variance s^2,

(a) $E(\bar{x}) = \mu$.

(b) $E((\bar{x} - \mu)^2) = \frac{\sigma^2}{n}$.

(c) $E(s^2) = \frac{n-1}{n}\sigma^2$.

The last part is independent of the above.

(d) Let X be a random variable with density function $f(x)$. Show, using elementary calculus, that the function

$$\phi(a) = E((X-a)^2)$$

takes its minimum value when $a = \mu(X)$, and in that case $\phi(a) = V(X)$.

Problem 4: Manipulation

$10 = 5 + 5$ pts

A number U is chosen at random in the interval $[0, 1]$.

(a) Find the probability that $T = \sin(\pi U) < \frac{1}{2}$.

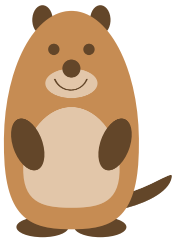
(b) Let X be a random variable uniformly distributed over $[c, d]$. For what choice of a and b do we have $U = aX + b$?

Problem 5: Educational attainment

10 = 5 + 5 pts

In August 2020, the Animal Bank released the Sharing Higher Education's Promise Report, highlighting the rising demand and supply of tertiary education. The Marmot Kingdom saw the fastest growth in its tertiary gross enrollment ratio (GER) during 2019 - 2020.

Back to 10 years ago, on the average, only 1 marmot in 1000 had a Bachelor degree.



(a) Find the probability that, in a county of 10,000 marmots, no one had a Bachelor degree.

(b) How many marmots would have to be surveyed to give a probability greater than $\frac{1}{2}$ of finding at least one marmot with a Bachelor degree?

Problem 6: Cupid's Arrow

15 = 7 + 8 pts

It's said that Cupid carries two kinds of arrows, one with a sharp golden point, and the other with a blunt tip of lead. A person wounded by the golden arrow is filled with uncontrollable desire, but the one struck by the lead feels aversion and desires only to flee.

When Apollo taunts Cupid as the lesser archer, Cupid shoots him with the golden arrow, but strikes the object of his desire, the nymph Daphne, with the lead.

The duration of an arrow is the length of time that particular arrow is effective. The duration of the golden arrow has an exponential density with average time 200 years. And that of the lead arrow also has an exponential density but with average time 500 years.

golden arrow

lead arrow

Apollo wonders what is the probability that

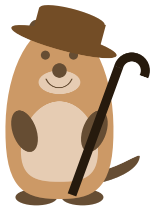
(a) Daphne will break free from the enchantment first?

(b) himself will break free from the enchantment first and Daphne has to wait another 300 years or more before her arrow wears off?

Problem 7: Man with No Name: A fistful of Nuts

15 = 7 + 8 pts

An unnamed stranger arrives at the little town of San Pecan on the Marmot-Squirrel border. He stays at Mr. Eyes closed's inn and befriends the aging innkeeper.



the Stranger

Mr. Eyes Closed

After settling down, the young and mysterious stranger opens a casino on the lobby.

A game is played as follows: a participant choose a random number X uniformly from $[0, 1]$. Then a sequence $Y_1, Y_2, \cdots$ of random numbers is chosen independently and uniformly from $[0, 1]$. The game ends the first time that $Y_i > X$ and the participant is paid $(i - 1)$ pecans.

(a) Let $P(N = n|X = x)$ be the probability that the payout will be $N = n$ pecans given $X = x$. Which amount would the participant be most likely to get given $X = x$?

Hint from Mr. Eyes Closed: As is mentioned in class, **most likely** can be interpreted as ...

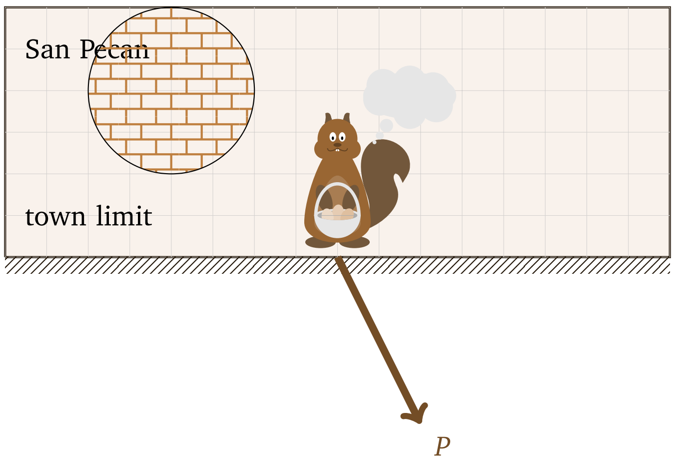
(b) What is a fair entrance fee for this game?

Problem 8: Man with No Name: Out of San Pecan

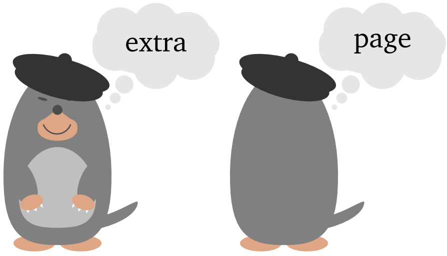
10 pts

Right across from the inn live the town sheriff Butter Beer, his wife Puff, and their son Nutella. As the only child in the family, Nutella was always under the umbrella of his parents and has never traveled far. However, after hearing about the college-going culture in the Marmot Kingdom, he decides to leave home and make his own way in the faraway places.

Now Nutella is standing on the town limit. He chooses, at random, a direction which will lead him to the world, and walks 2 miles in that direction. Let P denote his position. What is the expected distance from P to the town limit?



Hint from the stranger: Let θ be the angle ...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