Math 20: Probability
Homework 7

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Due Date: August 14, 2020

Please specify whether you complete the homework independently or cooperate with (get help from) the TA, your classmates, friends, online resources, etc.

For every problem, show the process and necessary details.

A pdf file is required for submission.

Problem 1

4 pts

Chapter 6.2 Exercise 15

Suppose that n people have their hats returned at random. Let $X_i = 1$ if the $i$the person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$. Then S_n is the total number of people who get their own hats back. Show that

(a) $E(X_i^2) = \frac{1}{n}$.

(b) $E(X_i \cdot X_j) = \frac{1}{n(n-1)}$ for $i \neq j$.

(c) $E(S_n^2) = 2$ (using (a) and (b)).

(d) $V(S_n) = 1$.

Problem 2

4 pts

Chapter 6.2 Exercise 20

We have two instruments that measure the distance between two points. The measurements given by the two instruments are random variables X_1 and X_2 that are independent with $E(X_1) = E(X_2) = \mu$, where μ is the true distance. From experience with instruments, we know the values of the variances σ_1^2 and σ_2^2. These variances are not necessarily the same. From two measurements, we estimate μ by the weighted average $\bar{\mu} = \omega X_1 + (1 - \omega)X_2$. Here ω is chosen in $[0, 1]$ to minimize the variance of $\bar{\mu}$.

(a) What is $E(\bar{\mu})$?

(b) How should ω be chosen in $[0, 1]$ to minimize the variance of $\bar{\mu}$?

Problem 3

4 pts

Chapter 6.2 Exercise 22

Let X and Y be two random variables defined on the finite sample space Ω. Assume that X, Y, $X + Y$, and $X - Y$ all have the same distribution. Prove that $P(X = Y = 0) = 1$.

Problem 4

4 pts

Chapter 6.3 Exercise 3

The lifetime, measure in hours, of the ACME super light bulb is a random variable T with density function $f_T(t) = \lambda^2 t e^{-\lambda t}$, where $\lambda = 0.05$. What is the expected lifetime of this light bulb? What is its variance?

Problem 5

4 pts

Chapter 6.3 Exercise 8

Let X be a random variable with mean μ and variance σ^2. Let $Y = aX^2 + bX + c$. Find the expected value of Y.

Problem 6

5 pts

Chapter 6.3 Exercise 10

Let X and Y be independent random variables with uniform density functions on $[0, 1]$. Find

(a) $E(|X - Y|)$.

(b) $E(\max(X, Y))$.

(c) $E(\min(X, Y))$.

(d) $E(X^2 + Y^2)$.

(e) $E((X + Y)^2)$.

Note: In previous homework sets, we have figured out the density functions for $|X - Y|$, $\max(X, Y)$, and $\min(X, Y)$. For instance, the density function for $\min(X, Y)$ is $f(x) = 2 - 2x$ on $[0, 1]$.

Problem 7

4 pts

Chapter 6.3 Exercise 12

Find $E(X^Y)$, where X and Y are independent random variables which are uniform on $[0, 1]$.

Note: No simulation needed.