Math 20: Probability

Homework 7

August 13, 2020

Problem 1

4 pts

Chapter 6.2 Exercise 15

Suppose that n people have their hats returned at random. Let $X_i = 1$ if the *i*the person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^n X_i$. Then S_n is the total number of people who get their own hats back. Show that (a) $E(X_i^2) = \frac{1}{n}$.

 $1 \ \mathrm{pts}$

We know that the distribution function of X_i is

$$m(X_i) = \begin{cases} \frac{1}{n}, & X_i = 1\\ 1 - \frac{1}{n}, & X_i = 0 \end{cases}$$

Hence

$$E(X_i^2) = 1^2 \times \frac{1}{n} + 0^2 \times (1 - \frac{1}{n}) = \frac{1}{n}.$$

(b) $E(X_i \cdot X_j) = \frac{1}{n(n-1)}$ for $i \neq j$.

$1 \ \mathrm{pts}$

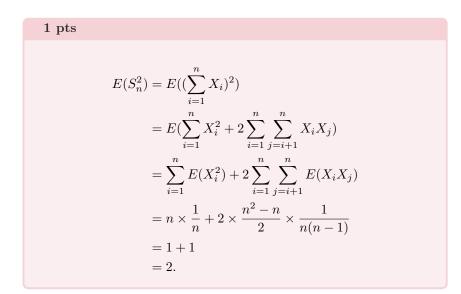
We know that only when $X_1 = X_2 = 1$ (both the *i*th and the *j*th persons get their hats back), $X_i \cdot X_j = 1$. Otherwise, the product is 0.

And the probability of getting $X_i \cdot X_j = 1$ is $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

Hence

$$E(X_i \cdot X_j) = 1 \times \frac{1}{n(n-1)} + 0 \times (1 - \frac{1}{n(n-1)}) = \frac{1}{n(n-1)}.$$

(c) $E(S_n^2) = 2$ (using (a) and (b)).



(d) $V(S_n) = 1$.

 $1 \ \mathrm{pts}$

We know that $E(S_n) = E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i) = 1.$ Therefore, $V(S_n) = E(S_n^2) - E^2(S_n)$ = 2 - 1= 1.

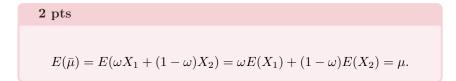
Problem 2

4 pts

Chapter 6.2 Exercise 20

We have two instruments that measure the distance between two points. The measurements given by the two instruments are random variables X_1 and X_2 that are independent with $E(X_1) = E(X_2) = \mu$, where μ is the true distance. From experience with instruments, we know the values of the variances σ_1^2 and σ_2^2 . These variances are not necessarily the same. From two measurements, we estimate μ by the weighted average $\bar{\mu} = \omega X_1 + (1 - \omega)X_2$. Here ω is chosen in [0, 1] to minimize the variance of $\bar{\mu}$.

(a) What is $E(\bar{\mu})$?



(b) How should ω be chosen in [0, 1] to minimize the variance of $\bar{\mu}$?

$$V(\bar{\mu}) = V(\omega X_1 + (1 - \omega)X_2)$$

= $\omega^2 V(X_1) + (1 - \omega)^2 V(X_2)$
= $\omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2$.

And the derivative with respect to ω is

$$\frac{d}{d\omega}V(\bar{\mu}) = 2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2.$$

When $\frac{d}{d\omega}V(\bar{\mu}) = 0$, we have

$$\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

Problem 3

4 pts

Chapter 6.2 Exercise 22

Let X and Y be two random variables defined on the finite sample space Ω . Assume that X, Y, X + Y, and X - Y all have the same distribution. Prove that P(X = Y = 0) = 1.

We know that if two random variables have the same distribution, they share the same expected value and variance.

Therefore, we have

$$E(X) = E(Y) = E(X + Y) = E(X) + E(Y).$$

Hence, we get that

$$E(X) = E(Y) = 0.$$

Further, we also have

$$V(X + Y) = V(X) + V(Y) + 2COV(X, Y) = V(X - Y) = V(X) + V(Y) - 2COV(X, Y).$$

Thus, we get that

$$COV(X,Y) = 0.$$

 As

$$V(X) = V(Y) = V(X+Y) = V(X) + V(Y) + 2COV(X,Y) = V(X) + V(Y),$$

we know that

$$V(X) = V(Y) = 0.$$

Given that E(X) = E(Y) = 0 and that V(X) = V(Y), we conclude that X = Y = 0. That is, P(X = Y = 0) = 1.

Problem 4

4 pts

Chapter 6.3 Exercise 3

The lifetime, measure in hours, of the ACME super light bulb is a random variable T with density function $f_T(t) = \lambda^2 t e^{-\lambda t}$, where $\lambda = 0.05$. What is the expected lifetime of this light bulb? What is its variance?

$2 \,\, \mathrm{pts}$

The expected value is

$$E(T) = \int_0^{+\infty} t f_T(t) dt$$

= $\int_0^{+\infty} \lambda^2 t^2 e^{-\lambda t} dt$
= $\frac{1}{\lambda} \int_0^{+\infty} u^2 e^{-u} du$
= $\frac{2}{\lambda}$
= 40.

2 pts

The variance is

$$V(T) = \int_{0}^{+\infty} t^{2} f_{T}(t) dt - E^{2}(T)$$

= $\int_{0}^{+\infty} \lambda^{2} t^{3} e^{-\lambda t} dt - E^{2}(T)$
= $\frac{1}{\lambda^{2}} \int_{0}^{+\infty} u^{3} e^{-u} du - E^{2}(T)$
= $\frac{6}{\lambda^{2}} - \frac{4}{\lambda^{2}}$
= $\frac{2}{\lambda^{2}}$
= 800.

Problem 5

4 pts

Chapter 6.3 Exercise 8

Let X be a random variable with mean μ and variance σ^2 . Let $Y = aX^2 + bX + c$. Find the expected value of Y.

$$\begin{split} E(Y) &= E(aX^2 + bX + c) \\ &= aE(X^2) + bE(X) + E(c) \\ &= a(V(X) + E^2(X)) + bE(X) + c \\ &= a(\sigma^2 + \mu^2) + b\mu + c. \end{split}$$

Problem 6

5 pts

Chapter 6.3 Exercise 10

Let X and Y be independent random variables with uniform density functions on [0,1]. Find

(a)
$$E(|X - Y|)$$
.

1 pts Given that f(x) = 2 - 2x on [0, 1], we have $E(|X - Y|) = \int_0^1 x(2 - 2x)dx = \frac{1}{3}.$

(b) $E(\max(X, Y))$.

1 pts

Given that f(x) = 2x on [0, 1], we have

$$E(|X - Y|) = \int_0^1 2x^2 dx = \frac{2}{3}$$

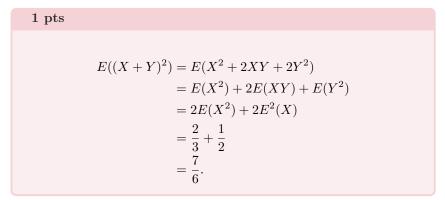
(c) $E(\min(X, Y))$.

Given that f(x)=2-2x on [0,1], we have $E(|X-Y|)=\int_0^1 x(2-2x)dx=\frac{1}{3}.$

(d) $E(X^2 + Y^2)$.

1 pts
$$E(X^2+Y^2)=E(X^2)+E(Y^2)=2E(X^2)=2\int_0^1 x^2 dx=\frac{2}{3}.$$

(e) $E((X+Y)^2)$.



Problem 7

4 pts

Chapter 6.3 Exercise 12

Find $E(X^Y)$, where X and Y are independent random variables which are uniform on [0, 1].

Note: No simulation needed.

$$\begin{split} E(X^Y) &= \int_0^1 \int_0^1 x^y dx dy \\ &= \int_0^1 (\frac{1}{y} x^{y+1} |_0^1) dy = \int_0^1 \frac{1}{y+1} dy \\ &= \ln(y+1) |_0^1 = \ln(2). \end{split}$$