Math 20: Probability

Homework 7

August 13, 2020

Problem 1

4 pts

Chapter 6.2 Exercise 15

Suppose that n people have their hats returned at random. Let $X_i = 1$ if the $i$the person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$. Then S_n is the total number of people who get their own hats back. Show that

(a) $E(X_i^2) = \frac{1}{n}$.

1 pts

We know that the distribution function of X_i is

$$m(X_i) = \begin{cases} \frac{1}{n}, & X_i = 1 \\ 1 - \frac{1}{n}. & X_i = 0 \end{cases}$$

Hence

$$E(X_i^2) = 1^2 \times \frac{1}{n} + 0^2 \times (1 - \frac{1}{n}) = \frac{1}{n}.$$

(b) $E(X_i \cdot X_j) = \frac{1}{n(n-1)}$ for $i \neq j$.

1 pts

We know that only when $X_1 = X_2 = 1$ (both the ith and the jth persons get their hats back), $X_i \cdot X_j = 1$. Otherwise, the product is 0.

And the probability of getting $X_i \cdot X_j = 1$ is $\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

Hence

$$E(X_i \cdot X_j) = 1 \times \frac{1}{n(n-1)} + 0 \times (1 - \frac{1}{n(n-1)}) = \frac{1}{n(n-1)}.$$

(c) $E(S_n^2) = 2$ (using (a) and (b)).

1 pts

$$\begin{aligned}
E(S_n^2) &= E((\sum_{i=1}^{n} X_i)^2) \\
&= E(\sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_i X_j) \\
&= \sum_{i=1}^{n} E(X_i^2) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} E(X_i X_j) \\
&= n \times \frac{1}{n} + 2 \times \frac{n^2 - n}{2} \times \frac{1}{n(n-1)} \\
&= 1 + 1 \\
&= 2.
\end{aligned}$$

(d) $V(S_n) = 1$.

1 pts

We know that $E(S_n) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 1$.

Therefore,

$$\begin{aligned} V(S_n) &= E(S_n^2) - E^2(S_n) \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

Problem 2

4 pts

Chapter 6.2 Exercise 20

We have two instruments that measure the distance between two points. The measurements given by the two instruments are random variables X_1 and X_2 that are independent with $E(X_1) = E(X_2) = \mu$, where μ is the true distance. From experience with instruments, we know the values of the variances σ_1^2 and σ_2^2. These variances are not necessarily the same. From two measurements, we estimate μ by the weighted average $\bar{\mu} = \omega X_1 + (1-\omega)X_2$. Here ω is chosen in $[0, 1]$ to minimize the variance of $\bar{\mu}$.

(a) What is $E(\bar{\mu})$?

2 pts

$$E(\bar{\mu}) = E(\omega X_1 + (1-\omega)X_2) = \omega E(X_1) + (1-\omega)E(X_2) = \mu.$$

(b) How should ω be chosen in $[0, 1]$ to minimize the variance of $\bar{\mu}$?

2 pts

$$\begin{aligned} V(\bar{\mu}) &= V(\omega X_1 + (1-\omega)X_2) \\ &= \omega^2 V(X_1) + (1-\omega)^2 V(X_2) \\ &= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2. \end{aligned}$$

And the derivative with respect to ω is

$$\frac{d}{d\omega}V(\bar{\mu}) = 2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2.$$

When $\frac{d}{d\omega}V(\bar{\mu}) = 0$, we have

$$\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

Problem 3

4 pts

Chapter 6.2 Exercise 22

Let X and Y be two random variables defined on the finite sample space Ω. Assume that X, Y, $X+Y$, and $X-Y$ all have the same distribution. Prove that $P(X = Y = 0) = 1$.

4 pts

We know that if two random variables have the same distribution, they share the same expected value and variance.

Therefore, we have

$$E(X) = E(Y) = E(X + Y) = E(X) + E(Y).$$

Hence, we get that

$$E(X) = E(Y) = 0.$$

Further, we also have

$$\begin{aligned} V(X+Y) &= V(X) + V(Y) + 2COV(X,Y) = \\ V(X-Y) &= V(X) + V(Y) - 2COV(X,Y). \end{aligned}$$

Thus, we get that

$$COV(X,Y) = 0.$$

As

$$V(X) = V(Y) = V(X+Y) = V(X)+V(Y)+2COV(X,Y) = V(X)+V(Y),$$

we know that

$$V(X) = V(Y) = 0.$$

Given that $E(X) = E(Y) = 0$ and that $V(X) = V(Y)$, we conclude that $X = Y = 0$. That is, $P(X = Y = 0) = 1$.

Problem 4

4 pts

Chapter 6.3 Exercise 3

The lifetime, measure in hours, of the ACME super light bulb is a random variable T with density function $f_T(t) = \lambda^2 t e^{-\lambda t}$, where $\lambda = 0.05$. What is the expected lifetime of this light bulb? What is its variance?

2 pts

The expected value is

$$\begin{aligned} E(T) &= \int_0^{+\infty} t f_T(t) dt \\ &= \int_0^{+\infty} \lambda^2 t^2 e^{-\lambda t} dt \\ &= \frac{1}{\lambda} \int_0^{+\infty} u^2 e^{-u} du \\ &= \frac{2}{\lambda} \\ &= 40. \end{aligned}$$

2 pts

The variance is

$$\begin{aligned} V(T) &= \int_0^{+\infty} t^2 f_T(t) dt - E^2(T) \\ &= \int_0^{+\infty} \lambda^2 t^3 e^{-\lambda t} dt - E^2(T) \\ &= \frac{1}{\lambda^2} \int_0^{+\infty} u^3 e^{-u} du - E^2(T) \\ &= \frac{6}{\lambda^2} - \frac{4}{\lambda^2} \\ &= \frac{2}{\lambda^2} \\ &= 800. \end{aligned}$$

Problem 5

4 pts

Chapter 6.3 Exercise 8

Let X be a random variable with mean μ and variance σ^2. Let $Y = aX^2 + bX + c$. Find the expected value of Y.

4 pts

$$
\begin{aligned}
E(Y) &= E(aX^2 + bX + c) \\
&= aE(X^2) + bE(X) + E(c) \\
&= a(V(X) + E^2(X)) + bE(X) + c \\
&= a(\sigma^2 + \mu^2) + b\mu + c.
\end{aligned}
$$

Problem 6

5 pts

Chapter 6.3 Exercise 10

Let X and Y be independent random variables with uniform density functions on $[0, 1]$. Find

(a) $E(|X - Y|)$.

1 pts

Given that $f(x) = 2 - 2x$ on $[0, 1]$, we have

$$E(|X - Y|) = \int_0^1 x(2 - 2x)dx = \frac{1}{3}.$$

(b) $E(\max(X, Y))$.

1 pts

Given that $f(x) = 2x$ on $[0, 1]$, we have

$$E(|X - Y|) = \int_0^1 2x^2 dx = \frac{2}{3}.$$

(c) $E(\min(X, Y))$.

1 pts

Given that $f(x) = 2 - 2x$ on $[0, 1]$, we have

$$E(|X - Y|) = \int_0^1 x(2 - 2x)dx = \frac{1}{3}.$$

(d) $E(X^2 + Y^2)$.

1 pts

$$E(X^2 + Y^2) = E(X^2) + E(Y^2) = 2E(X^2) = 2\int_0^1 x^2 dx = \frac{2}{3}.$$

(e) $E((X + Y)^2)$.

1 pts

$$\begin{aligned} E((X+Y)^2) &= E(X^2 + 2XY + 2Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) \\ &= 2E(X^2) + 2E^2(X) \\ &= \frac{2}{3} + \frac{1}{2} \\ &= \frac{7}{6}. \end{aligned}$$

Problem 7

4 pts

Chapter 6.3 Exercise 12

Find $E(X^Y)$, where X and Y are independent random variables which are uniform on $[0, 1]$.

Note: No simulation needed.

4 pts

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\begin{aligned}
E(X^Y) &= \int_0^1 \int_0^1 x^y dx dy \\
&= \int_0^1 (\frac{1}{y} x^{y+1}|_0^1) dy = \int_0^1 \frac{1}{y+1} dy \\
&= \ln(y+1)|_0^1 = \ln(2).
\end{aligned}
$$