Problem 1

Chapter 7.1 Exercise 5

Consider the following two experiments: the first has outcome $X$ taking on the values 0, 1, and 2 with equal probabilities; the second results in an (independent) outcome $Y$ taking on the value 3 with probability $\frac{1}{4}$ and 4 with probability $\frac{3}{4}$. Find the distribution of

(a) $Y + X$.

The distribution of $X$ is

$$p_X = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

And that of $Y$ is

$$p_Y = \begin{pmatrix} 3 & 4 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$
2 pts

The range of \( Y + X \) is \( \{3, 4, 5, 6\} \). We have

\[
P(Y + X = 3) = P(X = 0)P(Y = 3) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12},
\]

\[
P(Y+X = 4) = P(X = 0)P(Y = 4)+P(X = 1)P(Y = 3) = \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4} = \frac{1}{3},
\]

\[
P(Y+X = 5) = P(X = 1)P(Y = 4)+P(X = 2)P(Y = 3) = \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4} = \frac{1}{3},
\]

and

\[
P(Y + X = 6) = P(X = 2)P(Y = 4) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12},
\]

That is, the distribution of \( Y + X \) is

\[p_{Y+X} = \begin{pmatrix}
\frac{3}{12} & \frac{4}{12} & \frac{5}{12} & \frac{6}{12}
\end{pmatrix}.
\]

(b) \( Y - X \).

2 pts

The range of \( Y - X \) is \( \{1, 2, 3, 4\} \). We have

\[
P(Y - X = 1) = P(X = 2)P(Y = 3) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12},
\]

\[
P(Y-X = 2) = P(X = 1)P(Y = 3)+P(X = 2)P(Y = 4) = \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4} = \frac{1}{3},
\]

\[
P(Y- X = 3) = P(X = 0)P(Y = 3)+P(X = 1)P(Y = 4) = \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4} = \frac{1}{3},
\]

and

\[
P(Y - X = 4) = P(X = 0)P(Y = 4) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12},
\]

That is, the distribution of \( Y - X \) is

\[p_{Y-X} = \begin{pmatrix}
\frac{1}{12} & \frac{2}{12} & \frac{3}{12} & \frac{4}{12}
\end{pmatrix}.
\]
Problem 2

Chapter 7.2 Exercise 5

Suppose that \( X \) and \( Y \) are independent and \( Z = X + Y \). Find \( f_Z \) if

(a)

\[
\begin{align*}
    f_X(x) &= \begin{cases} 
        \lambda e^{-\lambda x}, & \text{if } x > 0 \\
        0, & \text{otherwise}
    \end{cases} \\
    f_Y(x) &= \begin{cases} 
        \mu e^{-\mu x}, & \text{if } x > 0 \\
        0, & \text{otherwise}
    \end{cases}
\end{align*}
\]

Assume that \( \lambda \neq \mu \).

2 pts

When \( Z = X + Y > 0 \),

\[
\begin{align*}
    f_Z(z) &= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy \\
    &= \int_{0}^{\infty} f_X(z - y) f_Y(y) dy \\
    &= \int_{0}^{z} f_X(z - y) f_Y(y) dy \\
    &= \int_{0}^{z} \lambda e^{-\lambda(z-y)} \mu e^{-\mu y} dy \\
    &= \lambda \mu e^{-\lambda z} \int_{0}^{z} e^{-(\mu-\lambda)y} dy \\
    &= \frac{\lambda \mu e^{-\lambda z}}{\mu - \lambda} (1 - e^{-(\mu-\lambda)z}) \\
    &= \frac{\lambda \mu}{\mu - \lambda} (e^{-\lambda z} - e^{-\mu z}).
\end{align*}
\]

Therefore,

\[
    f_Z(x) = \begin{cases} 
        \frac{\lambda \mu}{\mu - \lambda} (e^{-\lambda x} - e^{-\mu x}), & \text{if } x > 0 \\
        0, & \text{otherwise}
    \end{cases}
\]

(b)

\[
\begin{align*}
    f_X(x) &= \begin{cases} 
        \lambda e^{-\lambda x}, & \text{if } x > 0 \\
        0, & \text{otherwise}
    \end{cases}
\end{align*}
\]
\[ f_Y(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \]

2 pts

When \(0 < Z = X + Y \leq 1\),

\[
\begin{align*}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy \\
&= \int_{0}^{1} f_X(z - y) f_Y(y) dy \\
&= \int_{0}^{z} f_X(z - y) f_Y(y) dy \\
&= \int_{0}^{z} \lambda e^{-\lambda(z-y)} dy \\
&= \lambda e^{-\lambda z} \int_{0}^{z} e^{\lambda y} dy \\
&= e^{-\lambda z} (e^{\lambda z} - 1) \\
&= 1 - e^{-\lambda z}.
\end{align*}
\]

When \(Z > 1\),

\[
\begin{align*}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy \\
&= \int_{0}^{1} f_X(z - y) f_Y(y) dy \\
&= \int_{0}^{1} \lambda e^{-\lambda(z-y)} dy \\
&= \lambda e^{-\lambda z} \int_{0}^{1} e^{\lambda y} dy \\
&= e^{-\lambda z} (e^{\lambda} - 1) \\
&= e^{-\lambda(z-1)} - e^{-\lambda z}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
f_Z(x) &= \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x \leq 1 \\
e^{-\lambda(x-1)} - e^{-\lambda x}, & \text{if } x > 1 \\0, & \text{otherwise} \end{cases}
\end{align*}
\]
Problem 3

Chapter 7.2 Exercise 10

Let $X_1, X_2, \cdots, X_n$ be $n$ independent random variables each of which has an exponential density with mean $\mu$. Let $M$ be the minimum value of the $X_j$. Show that the density for $M$ is exponential with mean $\frac{\mu}{n}$.

**Hint**: Use cumulative distribution functions.

$$F_M(x) = P(M \leq x) = 1 - P(M > x) = 1 - P(\min(X_1, X_2, \cdots, X_n) > x) = 1 - P(X_1 > x)P(X_2 > x)\cdots P(X_n > x) = 1 - (e^{-x/\mu})^n = 1 - e^{-nx/\mu}.$$  

Therefore, 

$$f_m(x) = \frac{d}{dx} F_M(x) = \frac{d}{dx} (1 - e^{-nx/\mu}) = \frac{n}{\mu} e^{-nx/\mu}.$$  

That is to say, $M$ is exponential with mean $\frac{\mu}{n}$.

Problem 4

Chapter 8.2 Exercise 4

Let $X$ be a continuous random variable with values exponentially distributed over $[0, +\infty)$ with parameter $\lambda = 0.1$.

(a) Find the mean and variance of $X$.  


(b) Using Chebyshev’s Inequality, find an upper bound for the following probabilities: \( P(|X - 10| \geq 2), \ P(|X - 10| \geq 5), \ P(|X - 10| \geq 9), \) and \( P(|X - 10| \geq 20). \)

**Note:** Are the bounds all useful?

(c) Calculate these probabilities exactly, and compare with the bounds in (b).
Comparing these Chebyshev’s estimates with the exact values, we have:

\[(1, 0.852), (1, 0.617), (1, 0.245), (0.25, 0.0498)\].

**Problem 5**

3 pts

Chapter 8.2 Exercise 12

A share of common stock in the Pilsdorff beer company has a price \(Y_n\) on the \(n\)th business day of the year. Finn observes that the price change \(X_n = Y_{n+1} - Y_n\) appears to be a random variable with mean \(\mu = 0\) and variance \(\sigma^2 = \frac{1}{4}\). If \(Y_1 = 30\), find a lower bound for the following probabilities, under the assumption that the \(X_n\)’s are mutually independent.

(a) \(P(25 \leq Y_2 \leq 35)\).

1 pts

We know that \(Y_2 = Y_1 + (Y_2 - Y_1) = Y_1 + X_1\). Hence

\[
P(25 \leq Y_2 \leq 35) = P(-5 \leq X_1 \leq 5)
\]

\[
= P(|X_1| \leq 5) = 1 - P(|X_1| \geq 5)
\]

\[
\geq 1 - \frac{1/4}{5^2} = \frac{99}{100}.
\]
(b) \( P(25 \leq Y_{11} \leq 35) \).

1 pts

We know that

\[
Y_{11} = Y_1 + (Y_2 - Y_1) + (Y_3 - Y_2) + \cdots + (Y_{11} - Y_{10})
= Y_1 + X_1 + X_2 + \cdots + X_{10}.
\]

Hence

\[
P(25 \leq Y_{11} \leq 35) = P(-5 \leq X_1 + X_2 + \cdots + X_{10} \leq 5)
= P(|X_1 + X_2 + \cdots + X_{10}| \leq 5)
= 1 - P(|X_1 + X_2 + \cdots + X_{10}| \geq 5)
\geq 1 - \frac{10 \times 1/4}{5^2} = \frac{9}{10}.
\]

(c) \( P(25 \leq Y_{101} \leq 35) \).

1 pts

We know that

\[
Y_{101} = Y_1 + (Y_2 - Y_1) + (Y_3 - Y_2) + \cdots + (Y_{101} - Y_{100})
= Y_1 + X_1 + X_2 + \cdots + X_{100}.
\]

Hence

\[
P(25 \leq Y_{101} \leq 35) = P(-5 \leq X_1 + X_2 + \cdots + X_{100} \leq 5)
= P(|X_1 + X_2 + \cdots + X_{100}| \leq 5)
= 1 - P(|X_1 + X_2 + \cdots + X_{100}| \geq 5)
\geq 1 - \frac{100 \times 1/4}{5^2} = 0.
\]

Problem 6

4 pts

Chapter 9.1 Exercise 3
A true-false examination has 48 questions. June has probability $\frac{3}{4}$ of answering a question correctly. April just guesses on each question. A passing score is 30 or more correct answers. Compare the probability that June passes the exam with the probability that April passes it.

**Note:** You do not need to get a specific number. It is good enough to use the NA($a, b$) notation we have seen in class.

**2 pts**

For June, we need to consider the probability $i \leq S_n \leq j$ where $i = 30$ and $j = 48$ for a Binomial distribution with parameters $n = 48$ and $p = \frac{3}{4}$.

It is straightforward to see that $np = 36$, $npq = 9$ and $\sqrt{npq} = 3$. Further we obtain that

$$\frac{i - \frac{1}{2} - np}{\sqrt{npq}} = \frac{30 - \frac{1}{2} - 36}{3} = -\frac{13}{6},$$

and

$$\frac{j + \frac{1}{2} - np}{\sqrt{npq}} = \frac{48 + \frac{1}{2} - 36}{3} = -\frac{25}{6},$$

Therefore the probability is

$$P(30 \leq S_n \leq 48) \approx \text{NA}\left(-\frac{13}{6}, -\frac{25}{6}\right) \approx 0.985.$$
2 pts

For April, we need to consider the probability \( i \leq S_n \leq j \) where \( i = 30 \) and \( j = 48 \) for a Binomial distribution with parameters \( n = 48 \) and \( p = \frac{1}{2} \).

It is straightforward to see that \( np = 24, npq = 12 \) and \( \sqrt{npq} = \sqrt{12} \). Further we obtain that

\[
\frac{i - \frac{1}{2} - np}{\sqrt{npq}} = \frac{30 - \frac{1}{2} - 24}{\sqrt{12}} = \frac{11}{2\sqrt{12}}.
\]

and

\[
\frac{j + \frac{1}{2} - np}{\sqrt{npq}} = \frac{48 + \frac{1}{2} - 24}{\sqrt{12}} = \frac{49}{2\sqrt{12}}.
\]

Therefore the probability is

\[
P(30 \leq S_n \leq 48) \approx \text{NA}\left(\frac{11}{2\sqrt{12}}, \frac{49}{2\sqrt{12}}\right) \approx 0.056.
\]

Problem 7

5 pts

Chapter 9.2 Exercise 6

A bank accepts rolls of pennies and gives 50 cents credit to a customer without counting the contents. Assume that a roll contains 49 pennies 30 percent of the time, 50 pennies 60 percent of the time, and 51 pennies 10 percent of the time.

Note: You can simply use the \( \text{NA}(a, b) \) notation.

(a) Find the expected value and the variance for the amount that the bank loses on a typical roll.

1 pts

Let \( X \) be the amount that the bank loses on a typical roll.

\[
E(X) = 1 \times 30\% + 0 \times 60\% - 1 \times 10\% = 0.2.
\]

\[
V(X) = E(X^2) - E^2(X) = 1^2 \times 30\% + 0^2 \times 60\% + (-1)^2 \times 10\% - (0.2)^2 = 0.36.
\]
(b) Estimate the probability that the bank will lose more than 25 cents in 100 rolls.

1 pts

We need to consider the probability \( c \leq S_n \leq d \) where \( c = 26 \) and \( d = 100 \) for an independent trials process with \( n = 100 \).

It is straightforward to see that \( n\mu = 20 \), \( n\sigma^2 = 36 \) and \( \sqrt{n\sigma^2} = 6 \).

Further we obtain that

\[
\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{26 - 20}{6} = 1,
\]

and

\[
\frac{d - n\mu}{\sqrt{n\sigma^2}} = \frac{100 - 20}{6} = \frac{40}{3}.
\]

Therefore the probability is

\[
P(26 \leq S_n \leq 100) \approx \text{NA}(1, \frac{40}{3}) \approx 0.1587.
\]

(c) Estimate the probability that the bank will lose exactly 25 cents in 100 rolls.

1 pts

We need to consider the probability \( S_n = k \) where \( k = 25 \) for an independent trials process with \( n = 100 \).

We obtain that

\[
\frac{k - n\mu}{\sqrt{n\sigma^2}} = \frac{25 - 20}{6} = \frac{5}{6},
\]

Therefore the probability is

\[
P(S_n = 25) \approx \frac{1}{6} \phi\left(\frac{5}{6}\right) \approx 0.047.
\]

(d) Estimate the probability that the bank will lose any money in 100 rolls.
We need to consider the probability $c \leq S_n \leq d$ where $c = 1$ and $d = 100$ for an independent trials process with $n = 100$.

Similarly, we obtain that
\[
\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{1 - 20}{6} = -\frac{19}{6}.
\]

Therefore the probability is
\[
P(1 \leq S_n \leq 100) \approx \text{NA}\left(-\frac{19}{6}, \frac{40}{3}\right) \approx 0.999.
\]

(c) How many rolls does the bank need to collect to have a 99 percent chance of a net loss?

We need to consider the probability $c \leq S_n \leq d$ where $c = 1$ and $d = 100$ for an independent trials process with $n$ to be determined.

The probability is
\[
P(1 \leq S_n \leq n) \approx \text{NA}\left(\frac{1 - 0.2n}{0.6\sqrt{n}}, \frac{n - 0.2n}{0.6\sqrt{n}}\right).
\]

Notice that the upper bound is $\frac{n - 0.2n}{0.6\sqrt{n}} = \frac{4\sqrt{n}}{3}$. If $n$ is large enough (say $n \geq 10$), it can be considered approximately the same as $+\infty$ for the standard normal distribution. Therefore,
\[
P(1 \leq S_n \leq n) \approx \text{NA}\left(\frac{1 - 0.2n}{0.6\sqrt{n}}, +\infty\right) \geq 0.99.
\]

That is,
\[
\text{NA}\left(-\infty, \frac{1 - 0.2n}{0.6\sqrt{n}}\right) = 1 - \text{NA}\left(\frac{1 - 0.2n}{0.6\sqrt{n}}, +\infty\right) \leq 0.01.
\]

Referring to a $z$ table, we get
\[
\frac{1 - 0.2n}{0.6\sqrt{n}} \leq -2.33.
\]

And the smallest $n$ satisfying the above inequality is 59.
Problem 8

Chapter 9.3 Exercise 11

The price of one share of stock in the Pilsdorff Beer Company (see Problem 5) is given by \(Y_n\) on the \(n\)th day of the year. Finn observes that the differences \(X_n = Y_{n+1} - Y_n\) appears to be independent random variable with a common distribution having mean \(\mu = 0\) and variance \(\sigma^2 = \frac{1}{4}\). If \(Y_1 = 100\), estimate the probability that \(Y_{365}\) is

Note: For part (a), the answer can be obtained without using a calculator. For part (b) and (c), you can simply use the \(\text{NA}(a, b)\) notation.

(a) \(\geq 100\).

\[\begin{align*}
1 \text{ pts} \\
\text{Same as Problem 5, we can rewrite } Y_{365} \text{ as } Y_1 + \sum_{i=1}^{364} X_i. \\
\text{We need to consider the probability } c \leq S_n \leq d \text{ where } c = 0 \text{ and } d = +\infty \text{ for an independent trials process with } n = 364. \\
\text{It is straightforward to see that } n\mu = 0, \ n\sigma^2 = 91 \text{ and } \sqrt{n\sigma^2} = \sqrt{91}. \text{ Further we obtain that} \\
\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{0 - 0}{\sqrt{91}} = 0, \\
\text{and} \\
\frac{d - n\mu}{\sqrt{n\sigma^2}} = \frac{+\infty - 0}{\sqrt{91}} = \infty. \\
\text{Therefore the probability is} \\
P(0 \leq S_n \leq \infty) \approx \text{NA}(0, \infty) = 0.5.
\end{align*}\]

(b) \(\geq 110\).
We need to consider the probability \( c \leq S_n \leq d \) where \( c = 10 \) and \( d = +\infty \) for an independent trials process with \( n = 364 \).

Now we have

\[
\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{10 - 0}{\sqrt{91}} = \frac{10}{\sqrt{91}}.
\]

Therefore the probability is

\[
P(0 \leq S_n \leq \infty) \approx NA\left(\frac{10}{\sqrt{91}}, \infty\right) \approx 0.147.
\]

(c) \( \geq 120 \).

We need to consider the probability \( c \leq S_n \leq d \) where \( c = 20 \) and \( d = +\infty \) for an independent trials process with \( n = 364 \).

Now we have

\[
\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{20 - 0}{\sqrt{91}} = \frac{20}{\sqrt{91}}.
\]

Therefore the probability is

\[
P(0 \leq S_n \leq \infty) \approx NA\left(\frac{20}{\sqrt{91}}, \infty\right) \approx 0.018.
\]