BASIC CONCEPTS OF DISCRETE PROBABILITY

Chapter 1

Xingru Chen
xingru.chen.gr@dartmouth.edu

The theory of probability had its origins in games of chance and gambling.

French Society in the 1650's

- Gambling was popular and fashionable.

- Not restricted by law.

- As the games became more complicated and the stakes became larger there was a need for mathematical methods for computing chances.

Enter the Mathematicians

Gamblers in the 1717 France were used to bet on the event of getting at least one 1 (ace) in four rolls of a dice. As a more trying variation, two die were rolled 24 times with a bet on having at least one double ace. According to the reasoning of Chevalier de Méré, two aces in two rolls are 1/6 as likely as 1 ace in one roll. To compensate, de Méré thought, the two die should be rolled 6 times. And to achieve the probability of 1 ace in four rolls, the number of the rolls should be increased four fold - to 24. Thus reasoned Chevalier de Méré who expected a couple of aces to turn up in 24 double rolls with the frequency of an ace in 4 single rolls. However, he lost consistently.



Gambler

A well-known gambler, the chevalier De Mere

Mathematician

consulted Blaise Pascal in Paris about some questions about some games of chance.

Mathematician

Pascal began to correspond with his friend Pierre Fermat about these problems.

French Society in the 1650's

- Gambling was popular and fashionable.
- Not restricted by law.
- As the games became more complicated and the stakes became larger there was a need for mathematical methods for computing chances.

Enter the Mathematicians

Gambler

A well-known gambler, the chevalier De Mere

Mathematician

consulted Blaise Pascal in Paris about some questions about some games of chance.

Mathematician

Pascal began to correspond with his friend Pierre Fermat about these problems.

Classical Probability

probability

The correspondence between Pascal and Fermat is the origin of the mathematical study of probability.

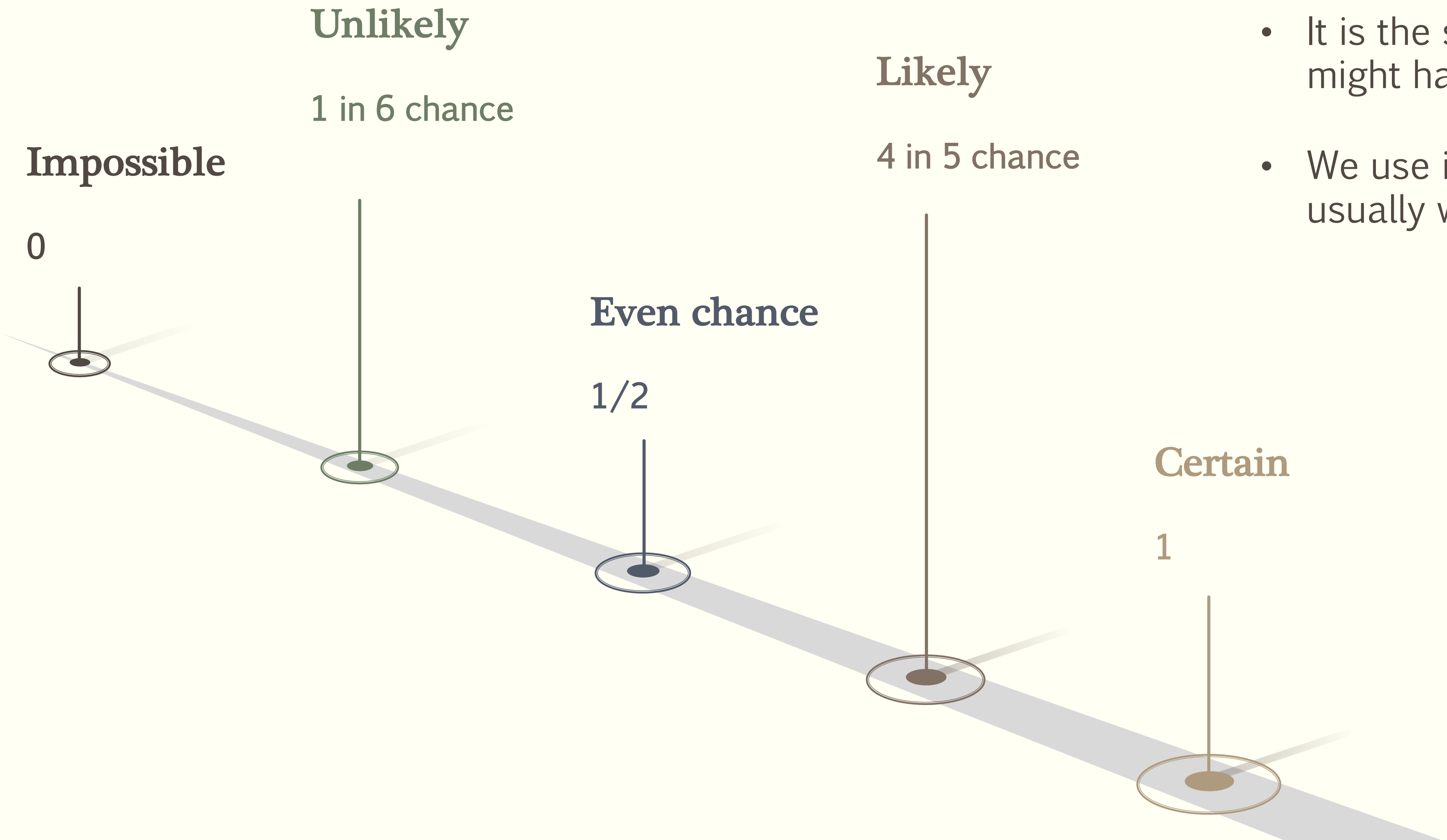
classical approach

The method they developed is now called the classical approach to computing probabilities.

equally likely outcomes

Suppose a game has n equally likely outcomes, of which m outcomes correspond to winning. Then the probability of winning is m/n.

Real Life Examples of Probability



- Probability has something to do with a chance.
- It is the study of things that might happen or might not.
- We use it most of the time, usually without thinking of it.

Questions for **chance of**

Clear Filters

Atheists: Suppose there is a zero chance of being caught—why *wouldn't* you cheat or steal if the Abrahamic God can't judge you?

Follow · 85

Is there any chances of re-examination of NEET 2017?

Follow · 304

What should Hillary Clinton be doing differently to maximize her chances of defeating Donald Trump?

Follow · 135

Press a button and there is a 99% chance of doubling your money and a 1% chance of losing it all. You are given $1 to start. How many times will you press the button?

Follow · 33

Do atheists think that they are clever by equating a god that has a lesser chance of existing to a God that has a higher chance of existing?

Follow · 7

What are my chances of getting PR in Canada ?

Follow · 162

What are the chances of Neet 2017 getting cancelled?

Follow · 101

What are chances of AAP winning Punjab elections?

Questions for **probability of**

Clear Filters

What is the probability of getting 53 Sundays in a year?

Follow · 58

If three coins are tossed simultaneously, what is the probability of getting at least two heads?

Follow · 87

If P(E) =0.03, what is the probability of 'not E'?

Follow · 21

If two normal dice are thrown together, what is the probability of getting a sum of 7?

Follow · 48

The probability of a yellow taxi is 0.2. The probability of a Fiat taxi is 0.4. The probability of a yellow or Fiat taxi is 0.3. What's the probability of hiring a taxi that is not a yellow Fiat?

Follow · 17

What is the probability of being born?

Follow · 20

What's the probability that a leap year has 53 Sundays?

Follow · 73

Weather Forecasting

FRI MAY 29	Scattered Thunderstorms	**31°/19°**	50%	SSW 16 km/h	63%
	UV INDEX 6 of 10	SUNRISE 5:12 am	SUNSET 8:21 pm	MOONRISE 11:44 am	MOONSET 1:28 am
	Scattered showers and thunderstorms. High 31C. Winds SSW at 10 to 15 km/h. Chance of rain 50%.				
FRI NIGHT MAY 29	Scattered Thunderstorms	**--/19°**	60%	SSW 13 km/h	80%
	UV INDEX 0 of 10	SUNRISE 5:12 am	SUNSET 8:21 pm	MOONRISE 11:44 am	MOONSET 1:28 am
	Scattered showers and thunderstorms. Low 19C. Winds SSW at 10 to 15 km/h. Chance of rain 60%.				

Batting Average in Cricket

16 APR 2017 American College Cricket League, 2017 Season Northeastern Deadham Field - 1, Massachusetts

NORTHEASTERN UNIVERSITY
Northeastern Huskies
206/5 (20.0 ov)

DARTMOUTH COLLEGE
Dartmouth Big Green Cricket
200/10 (19.4 ov)

Northeastern University, Northeastern Huskies - Won by 6 runs

18 SEP 2016 American College Cricket League, 2016/2017 League Season Wicked Blue Field - 1, Massachusetts

UMASS LOWELL
UMass Lowell Riverhawks
221/6 (20.0 ov)

DARTMOUTH COLLEGE
Dartmouth Big Green Cricket
0/0 (0.0 ov)

Dartmouth College, Dartmouth Big Green Cricket - Won by 10 wickets

17 APR 2016 American College Cricket League, 2016 League Season Chase AstroTurf field (Dartmouth) - 1, Massachusetts

DARTMOUTH COLLEGE
Dartmouth Big Green Cricket
174/1 (17.2 ov)

UMASS LOWELL
UMass Lowell Riverhawks
170/6 (20.0 ov)

Dartmouth College, Dartmouth Big Green Cricket - Won by 9 wickets

10 APR 2016 American College Cricket League, 2016 League Season Chase AstroTurf field (Dartmouth) - 1, Massachusetts

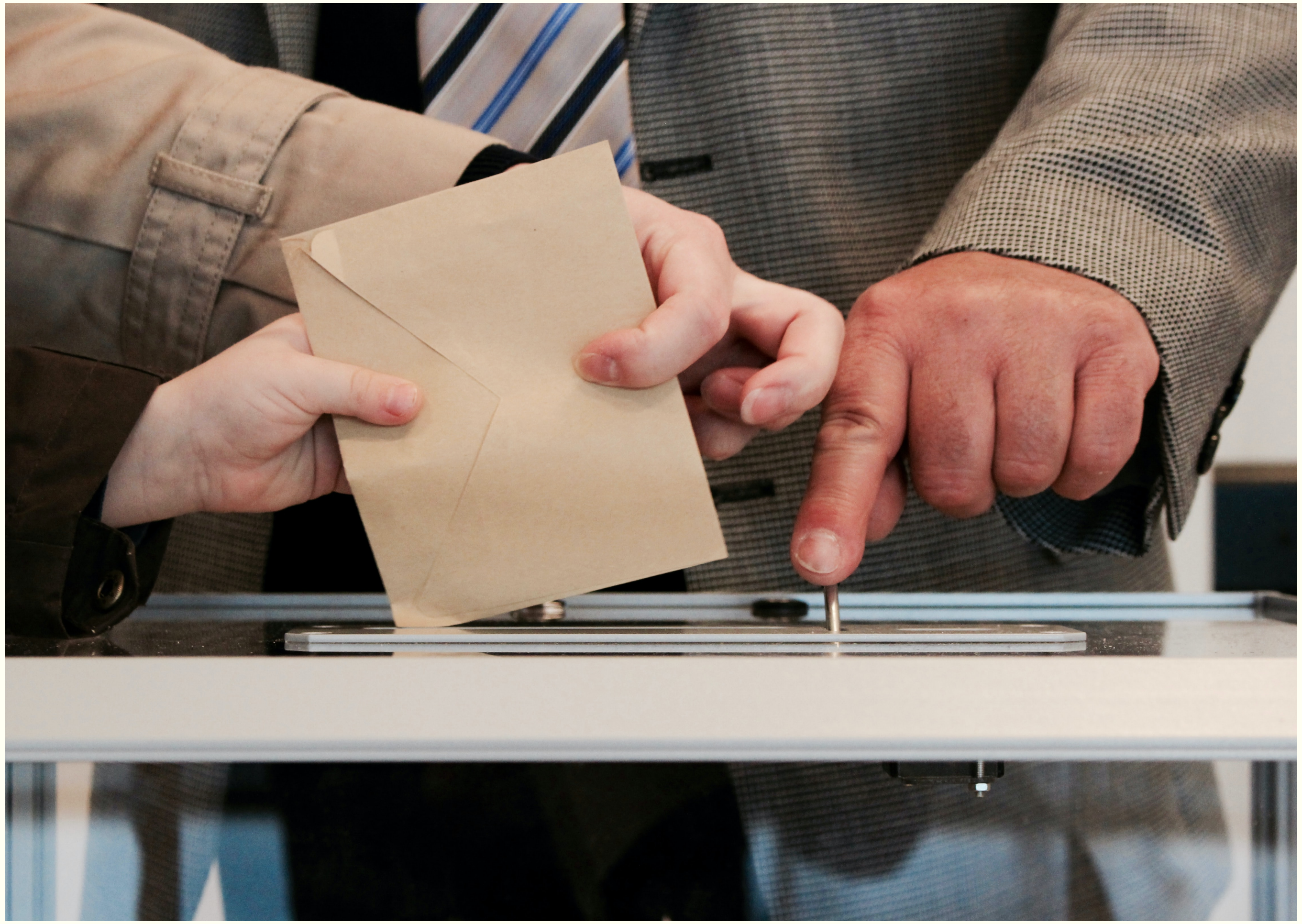
DARTMOUTH COLLEGE
Dartmouth Big Green Cricket
110/10 (12.4 ov)

NORTHEASTERN UNIVERSITY
Northeastern Huskies
244/5 (20.0 ov)

Northeastern University, Northeastern Huskies - Won by 134 runs

Politics

- Many politics analysts use the tactics of probability to predict the outcome of the election's results.
- For example, they may predict a certain political party to come into power based on the results of exit polls.



Insurance

- Insurance companies rely on the **Law of Large Numbers** to help estimate the value and frequency of future claims they will pay to policyholders.

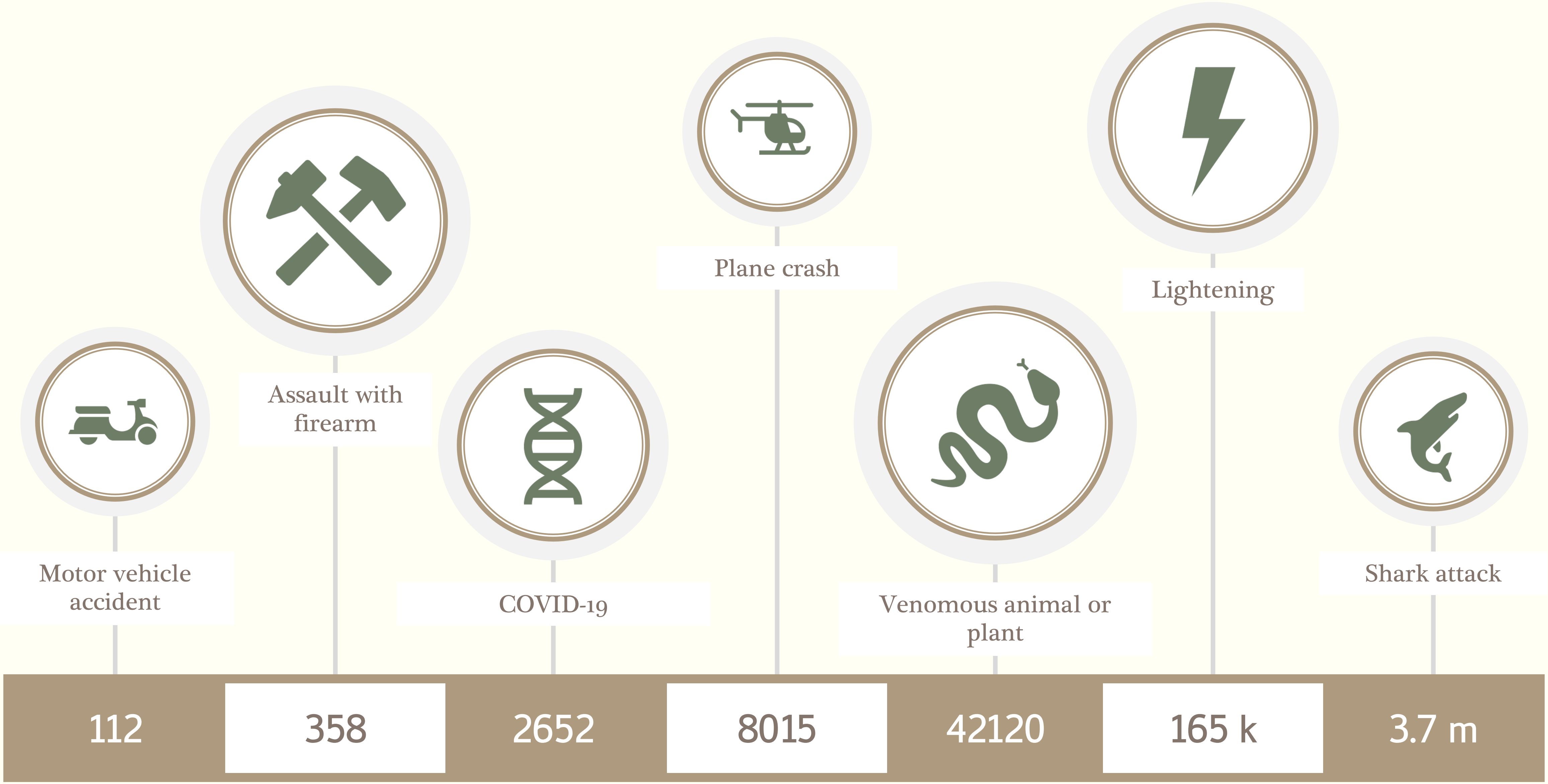
- When it works perfectly, insurance companies run a stable business, consumers pay a fair and accurate premium, and the entire financial system avoids serious disruption.

- However, the theoretical benefits from the law of large numbers do not always hold up in the real world.

LOTTERY

- In a typical Lottery game, each player chooses six distinct numbers from a particular range.
- If all the six numbers on a ticket match with that of the winning lottery ticket, the ticket holder is a Jackpot winner regardless of the order of the numbers.
- The probability of this happening is 1 out of 10 lakh (million).

Odds of dying in selected events in the United States: 1 in …



XC 2020

Odds of dying in selected events in the United States: 1 in …

COVID-19

2652

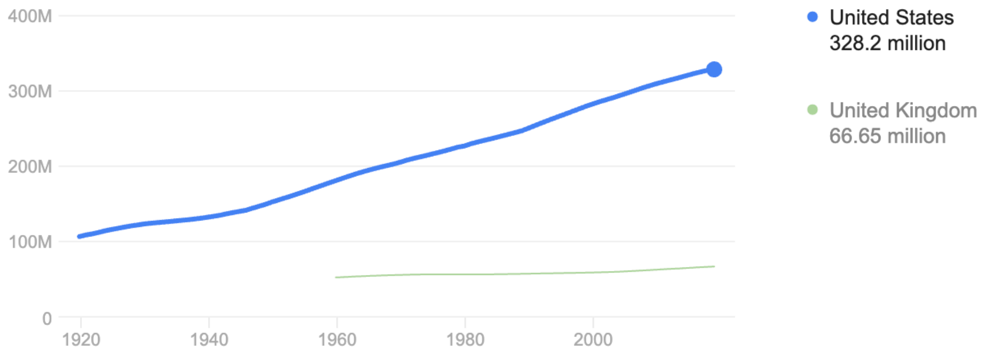
United States

Confirmed	Recovered	Deaths
2,427,448	747,316	123,751

Updated less than 1 hour ago • Source: Wikipedia

United States / Population

328.2 million (2019)



Explore more

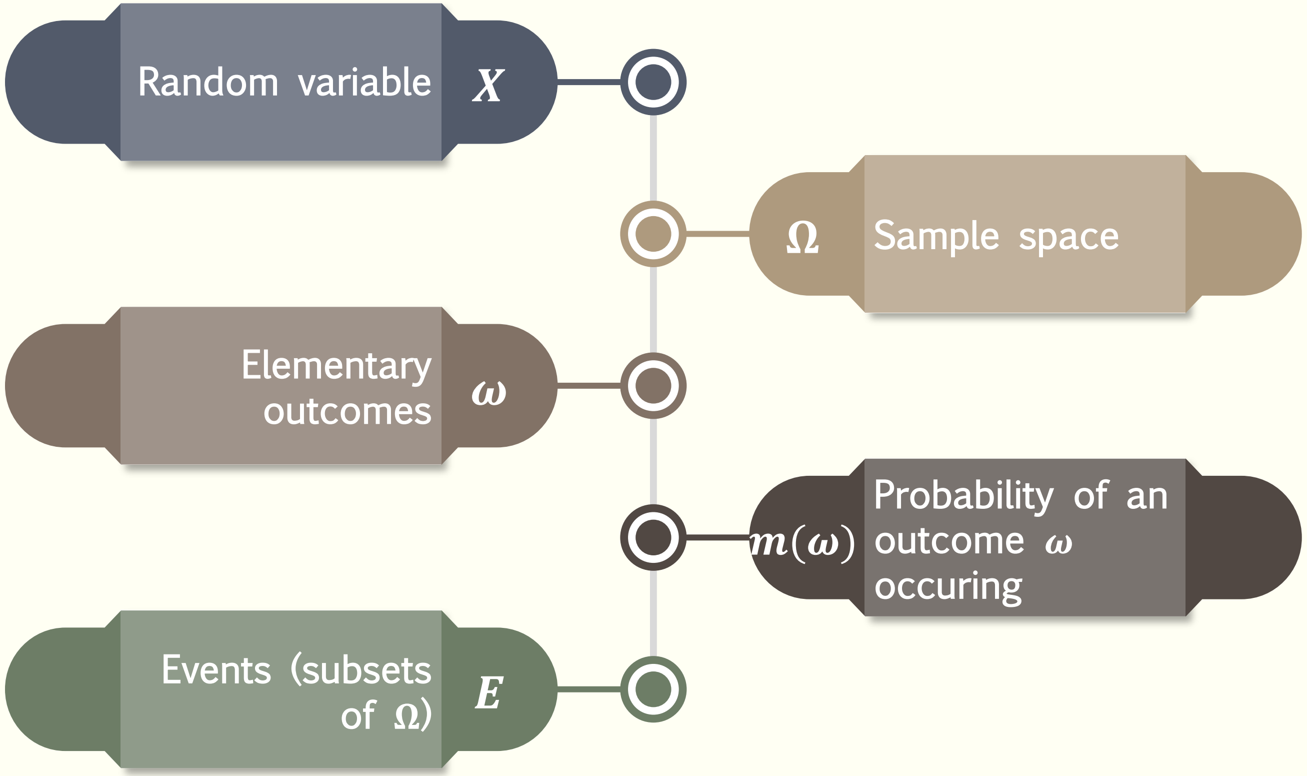
Sources include:United States Census Bureau, Eurostat

Feedback

Discrete Probability Distribution

- Random variable X
- Ω Sample space
- Elementary outcomes ω
- $m(\omega)$ Probability of an outcome ω occuring
- Events (subsets of Ω) E

Distribution Function of Random Variable

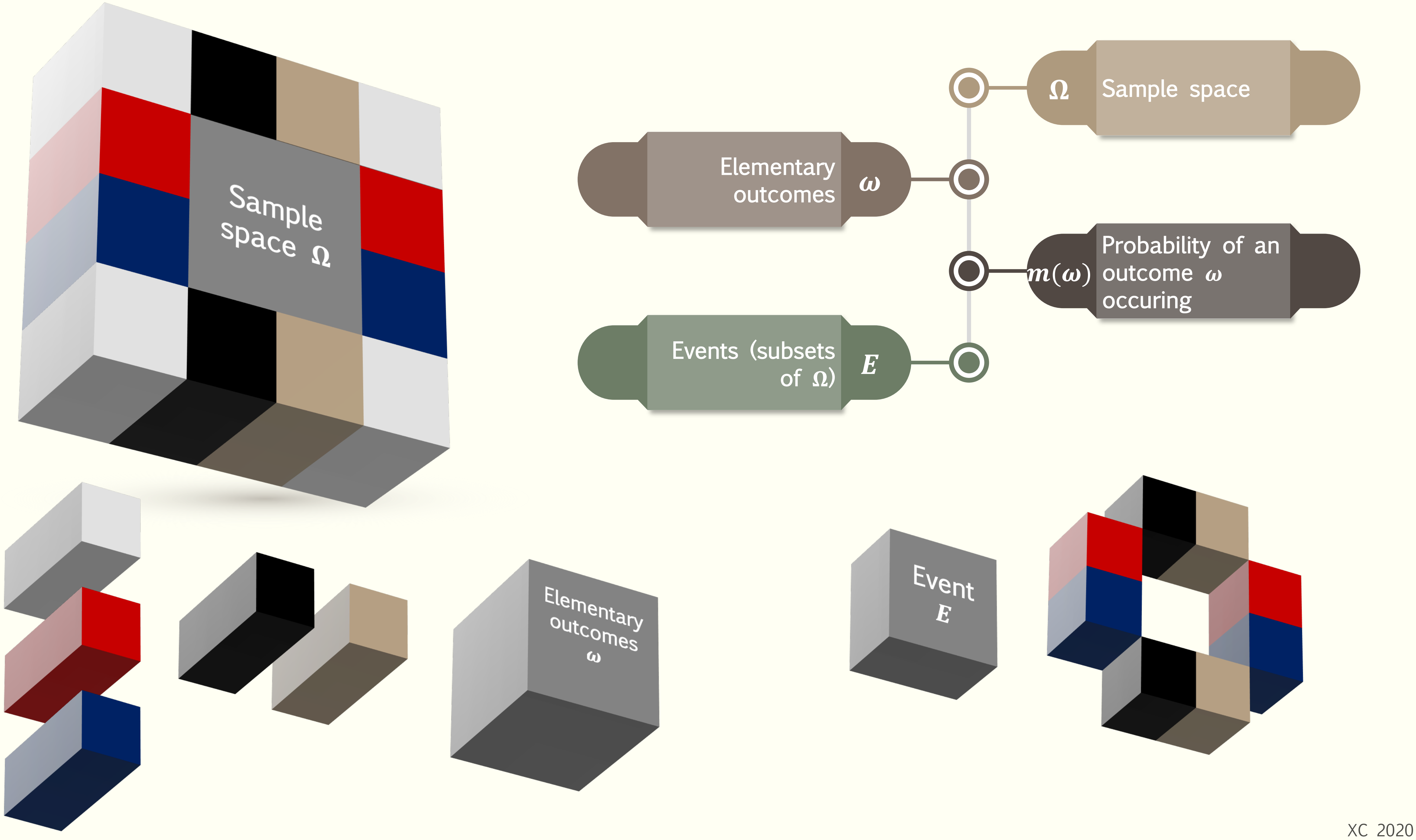


A distribution function for X is a real-valued function m whose domain is Ω and which satisfies:

- $m(\omega) \geq 0$, for all $\omega \in \Omega$, and
- $\sum_{\omega \in \Omega} m(\omega) = 1$.

For any subset E of Ω, we define the probability of E to be the number $P(E)$ given by

$$P(E) = \sum_{\omega \in E} m(\omega).$$


Sample space Ω
Elementary outcomes ω
Event E
Ω Sample space
Elementary outcomes ω
m(ω) Probability of an outcome ω occuring
Events (subsets of Ω) E
XC 2020

Finite Sample Space

Tossing a coin

$$\Omega = \{H, T\}$$

01

02

Rolling a die

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Tossing two coins

indistinguishable coins: $\Omega = \{HH, HT, TT\}$
distinct coins: $\Omega = \{HH, HT, TH, TT\}$

03

Toss a coin	
sample space	$\Omega = \{H, T\}$
elementary outcome	$\omega = H$ or T
probability distribution function	$m(\omega) = \frac{1}{2}$

Roll a dice	
sample space	$\Omega = \{1, 2, 3, 4, 5, 6\}$
elementary outcome	$\omega = 1, 2, 3, 4, 5$ or 6
probability distribution function	$m(\omega) = \frac{1}{6}$

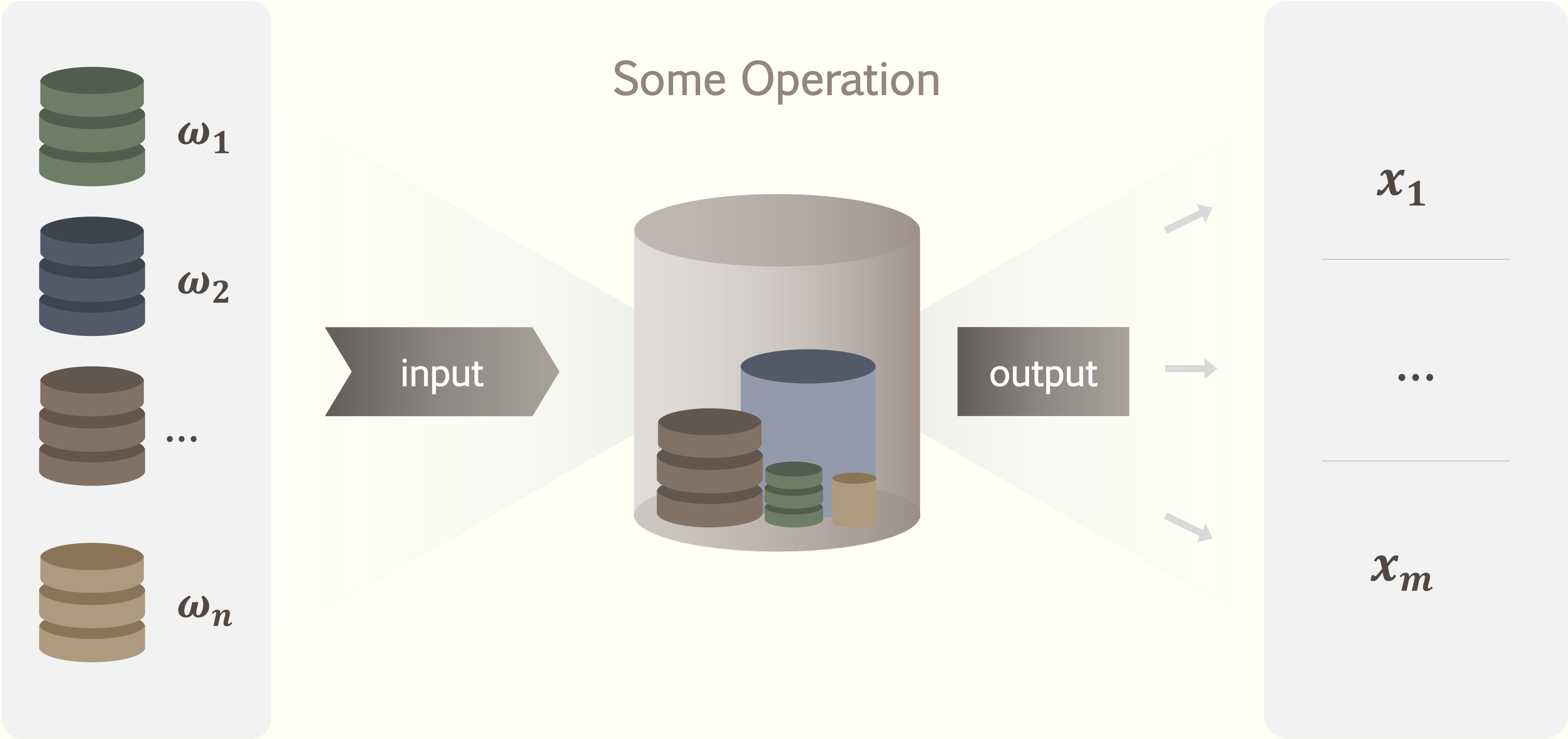
Toss two coins (distinct)	
sample space	$\Omega = \{HH, HT, TH, TT\}$
elementary outcome	$\omega = HH, HT, TH$ or TT
probability distribution function	$m(\omega) = \frac{1}{4}$

- A sample space is a collection of all possible outcomes of a random experiment.
- **A random variable is a function defined on a sample space.**
- The notation used for random variable is **an uppercase letter**. So if we have a random variable that maps sample space to real numbers, we have
$$X\colon \Omega \to \mathbb{R}$$
- If that random variable X is a set of possible values from a random experiment, then
$$X\colon \Omega \to \Omega$$

What is a function?

?

What Is a Function?



Toss a coin

sample space	$\Omega = \{H, T\}$
elementary outcome	$\omega = H$ or T
probability distribution function	$m(\omega) = \frac{1}{2}$
random variable	$X = \omega$
	$X = \begin{cases} 1, & \omega = H \\ 0, & \omega = T \end{cases}$
	$X = \begin{cases} \text{True}, & \omega = H \\ \text{False}, & \omega = T \end{cases}$
	$X = \begin{cases} 2, & \omega = H \\ 2, & \omega = T \end{cases}$
	…

Roll a dice	
sample space	$\Omega = \{1, 2, 3, 4, 5, 6\}$
elementary outcome	$\omega = 1, 2, 3, 4, 5$ or 6
probability distribution function	$m(\omega) = \frac{1}{6}$
random variable	$X = \omega$
	$X = \omega^2$
	$X = \begin{cases} 1, & \omega \text{ is even} \\ 0, & \omega \text{ is odd} \end{cases}$
	…

Toss two coins (distinguishable)

sample space	$\Omega = \{HH, HT, TH, TT\}$
elementary outcome	$\omega = HH, HT, TH$ or TT
probability distribution function	$m(\omega) = \frac{1}{4}$
random variable	$X = \omega$
	$X = \begin{cases} 1, & \omega \text{ has at least a Head} \\ 0, & \omega \text{ has no Head} \end{cases}$
	$X = \begin{cases} 1, & \omega \text{ has two same faces} \\ 0, & \omega \text{ has different faces} \end{cases}$
	…

Game	Tossing a coin	Rolling a dice	Tossing two coins
Ω	$\{H, T\}$	$\{1,2,3,4,5,6\}$	$\{HH, HT, TH, TT\}$
ω	H or T	$1,2,3,4,5$ or 6	HH, HT, TH or TT
$m(\omega)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$
E			
$P(E)$			

Game	Rolling a dice	Tossing two coins
Ω	$\{1,2,3,4,5,6\}$	$\{HH, HT, TH, TT\}$
$m(\omega)$	$\frac{1}{6}$	$\frac{1}{4}$
E	The number is • even. • odd. • no greater than 5. • a complete square.	• At least one Head. • The first one being Tail. • Two tosses yielding the same result.
$P(E)$	• $1/2$ • $1/2$ • $5/6$ • $1/3$	• $3/4$ • $1/2$ • $1/2$

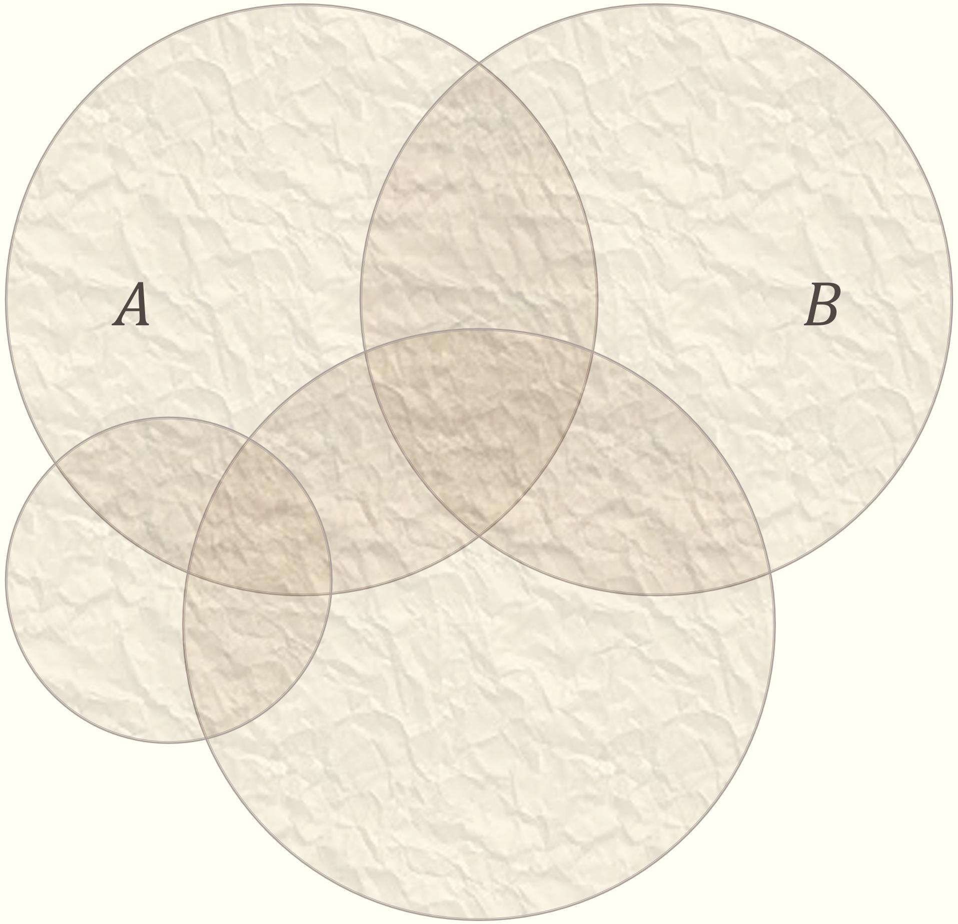
Complicated Events Described by Set Operations

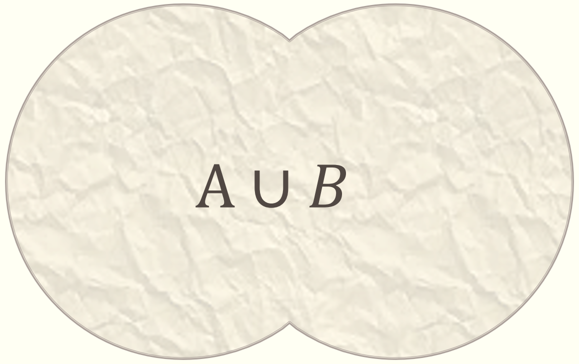
Let A and B be two sets.

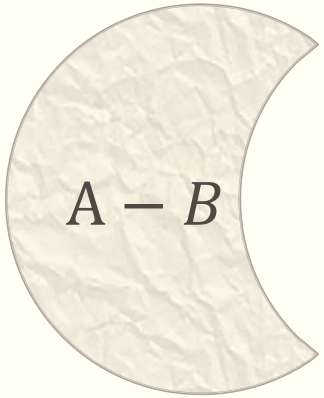
- The **union** of A and B is the set
  $$\mathrm{A} \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

- The **intersection** of A and B is the set
  $$\mathrm{A} \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- The **difference** of A and B is the set
  $$\mathrm{A} - B = \{x \mid x \in A \text{ and } x \notin B\}.$$



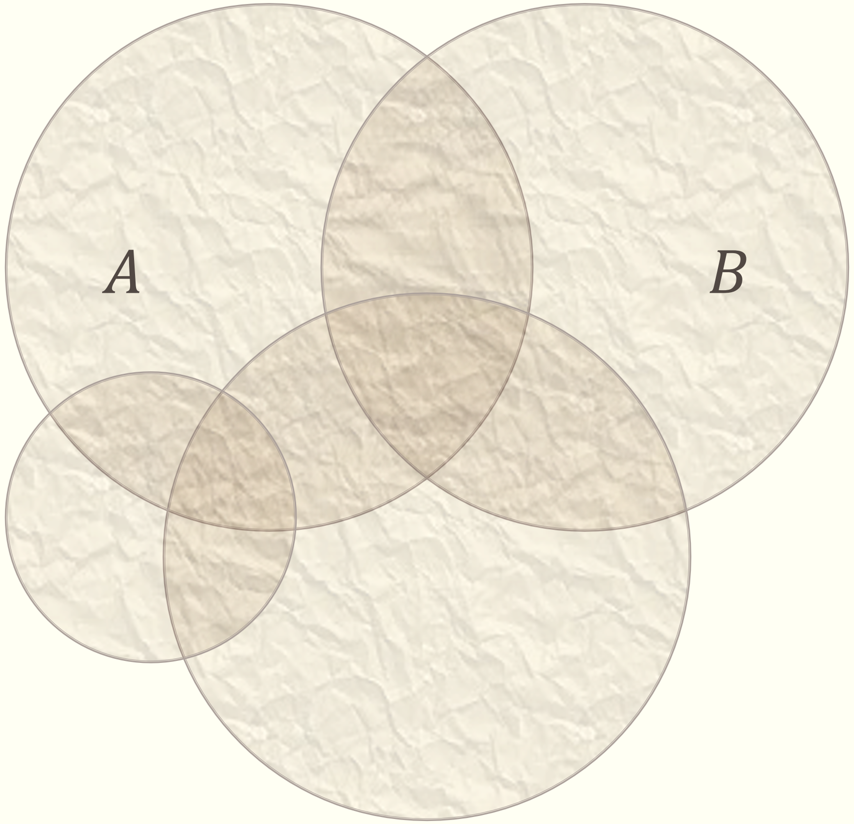




Complicated Events Described by Set Operations

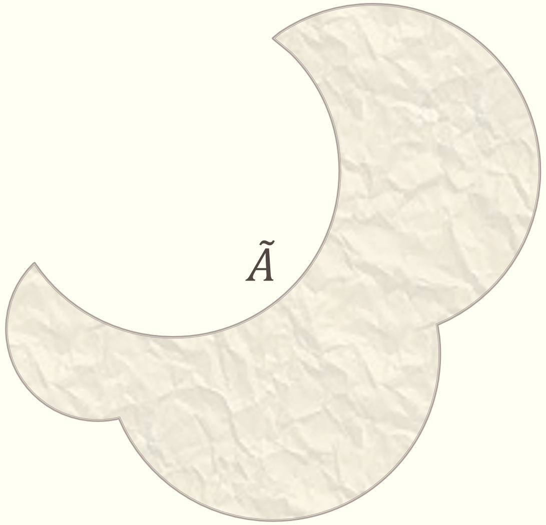
Let A and B be two sets.

- The set A is a subset of B, written $A \subset B$, if every element of A is also an element of B.
- The complement of A is the set
$$\tilde{A} = \{x \mid x \in \Omega \text{ and } x \notin A\}.$$



Is A a subset of B?

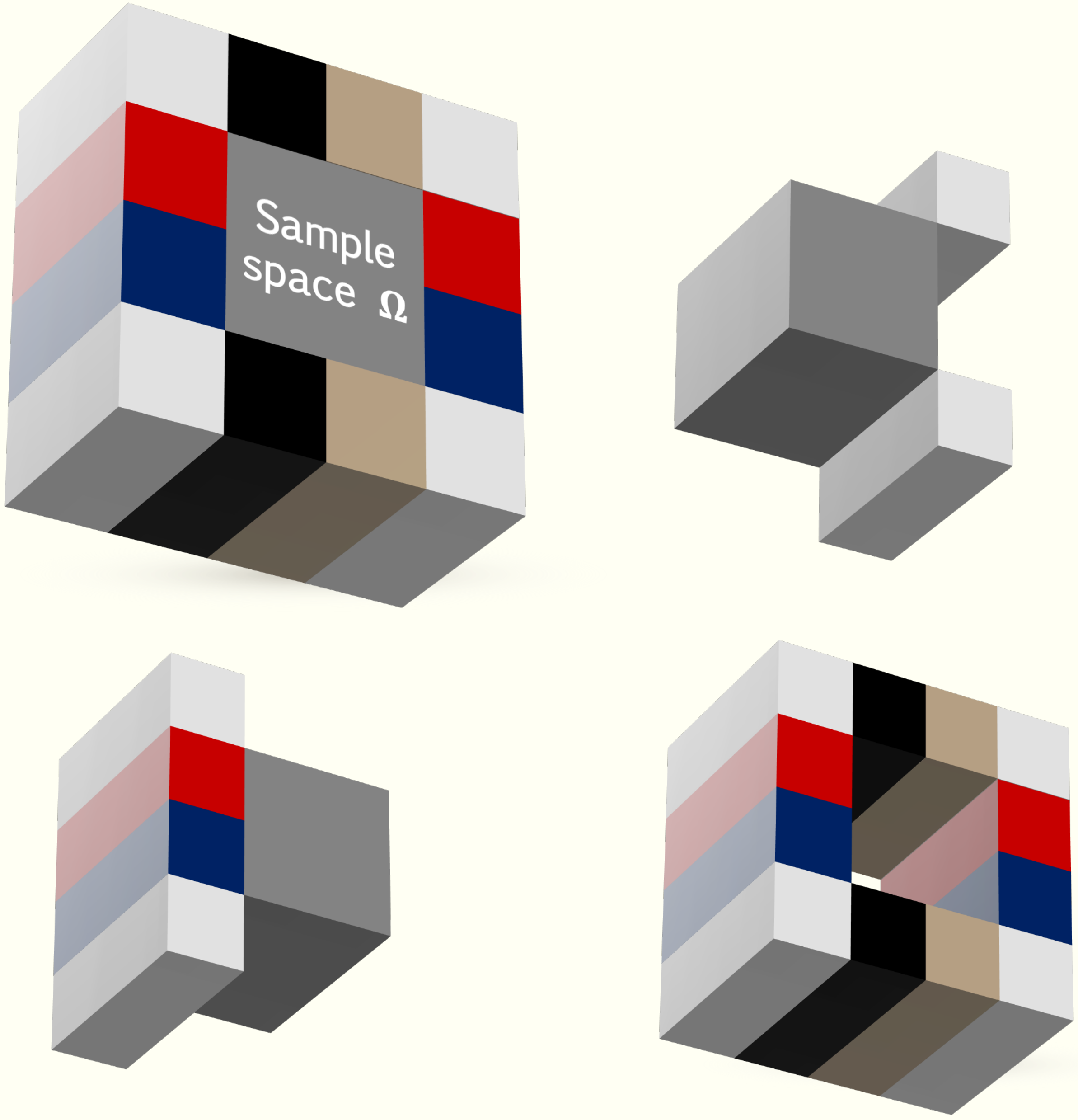
?



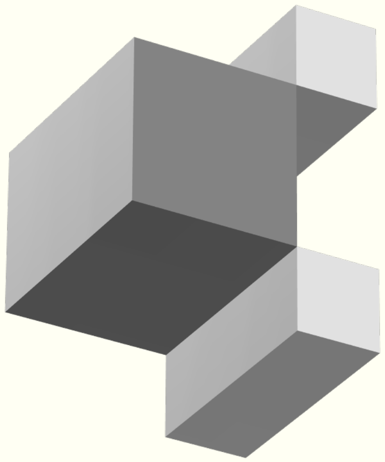
Important Properties

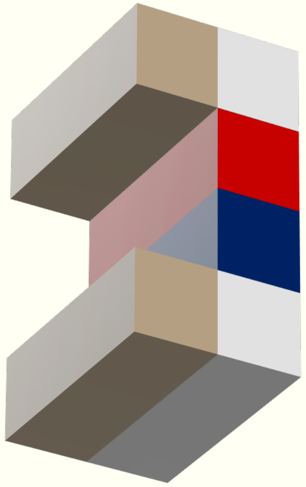
The probabilities assigned to events by a distribution on a sample space Ω satisfy the following properties:

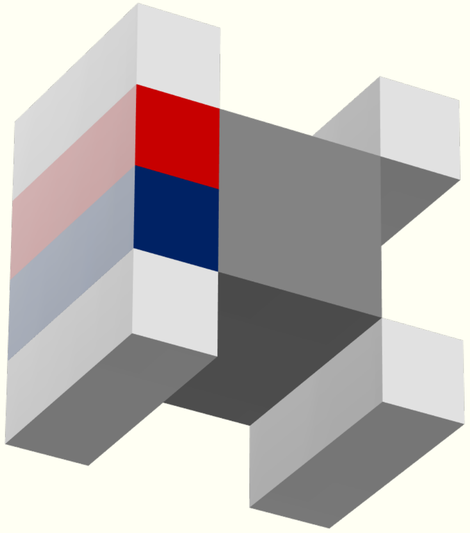
- $P(E) \geq 0$ for every $E \in \Omega$.
- $P(\Omega) = 1$.
- If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$.
- If A and B are disjoint subsets of Ω, then $P(A \cup B) = P(A) + P(B)$.
- $P(\tilde{A}) = 1 - P(A)$ for every $A \in \Omega$.

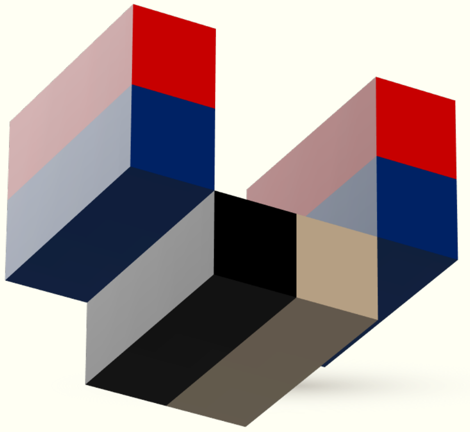


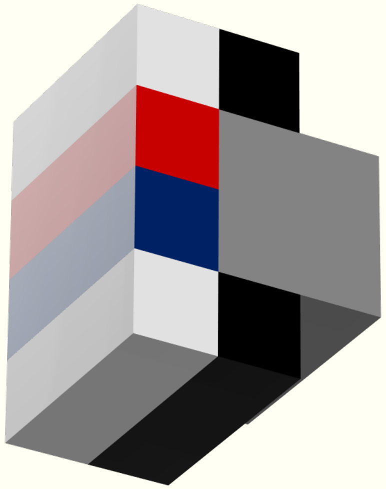
- If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$.
- If A and B are disjoint subsets of Ω, then $P(A \cup B) = P(A) + P(B)$.
- $P(\tilde{A}) = 1 - P(A)$ for every $A \in \Omega$.











Game		Rolling a dice	Tossing two coins
Ω		$\{1, 2, 3, 4, 5, 6\}$	$\{HH, HT, TH, TT\}$
$m(\omega)$		$\frac{1}{6}$	$\frac{1}{4}$
E		• The number is even. $E = \{2, 4, 6\}$	• Two tosses yielding the same result. $E = \{HH, TT\}$
F	$E \subset F$	$F = \{1,2,4,6\}$	$F = \{HH, TT\}$
	$E \cap F = \emptyset$	$F = \{1\}$	$F = \{HT\}$
$\tilde{E}$		$\tilde{E} = \{1,3,5\}$	$\tilde{E} = \{HT, TH\}$

Important Properties

- If $A_1, \dots, A_n$ are pairwise disjoint subsets of Ω (i.e., no two of the A_i have an element in common), then

$$P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i).$$

- If $A_1, \dots, A_n$ are pairwise disjoint subsets with $\Omega = A_1 \cup \cdots \cup A_n$, and let E be any event. Then

$$P(E) = \sum_{i=1}^{n} P(E \cap A_i).$$

- For any two events A and B,

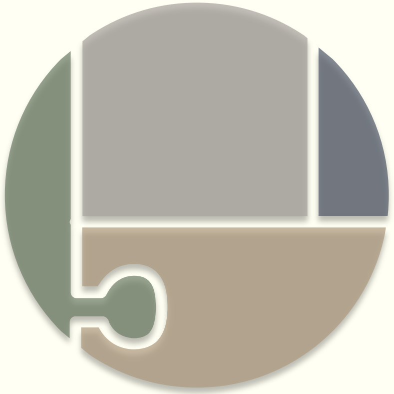
$$P(A) = P(A \cap B) + P(A \cap \tilde{B}).$$

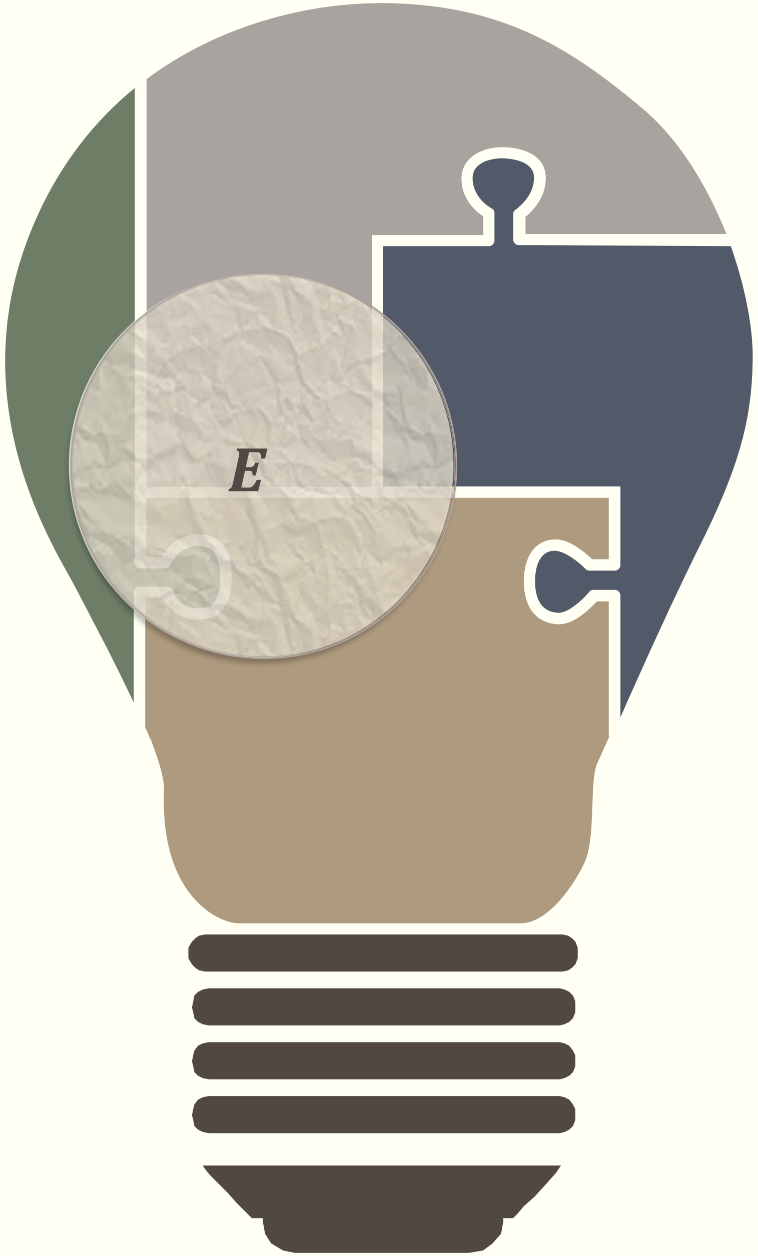
- If $A_1, \dots, A_n$ are pairwise disjoint subsets of Ω, then

$$P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i).$$

- If $A_1, \dots, A_n$ are pairwise disjoint subsets with $\Omega = A_1 \cup \cdots \cup A_n$, and let E be any event. Then

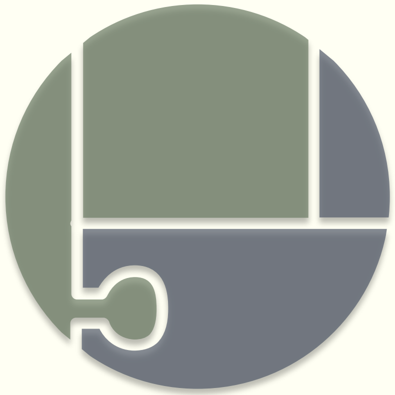
$$P(E) = \sum_{i=1}^{n} P(E \cap A_i).$$

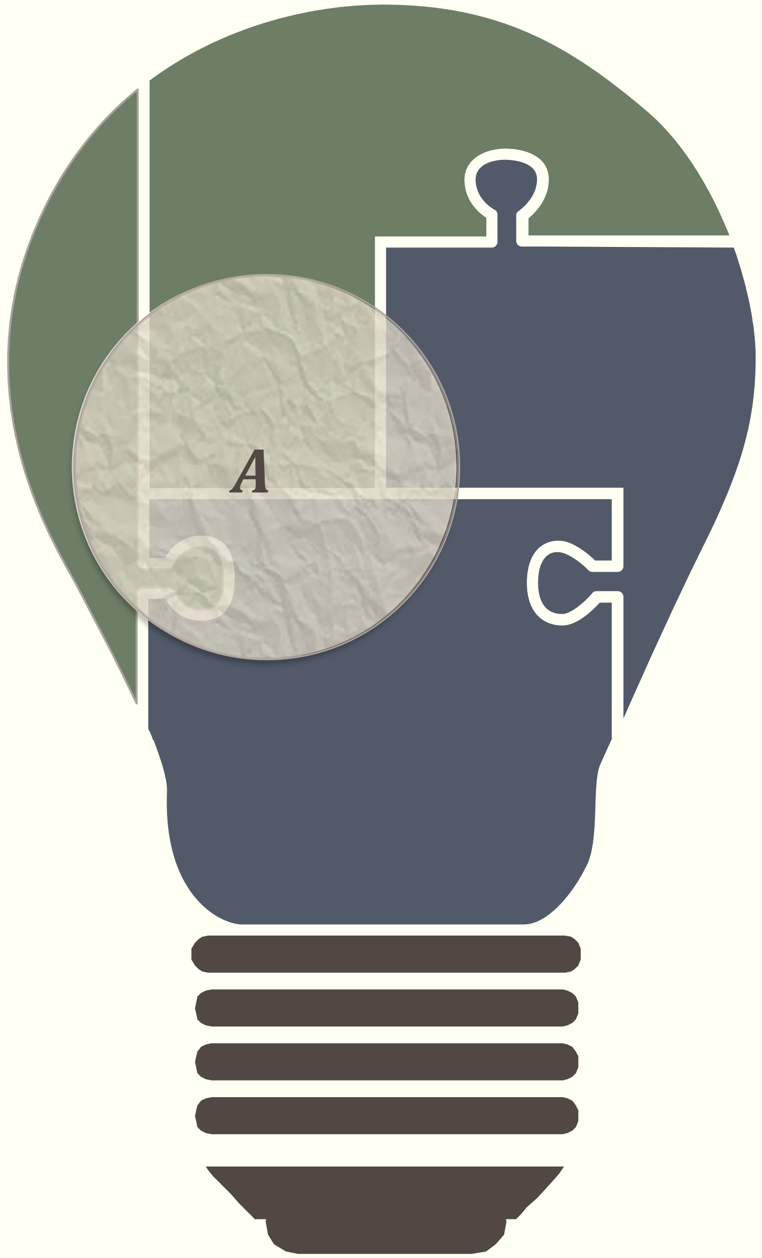




- For any two events A and B,

$$P(A) = P(A \cap B) + P(A \cap \tilde{B}).$$





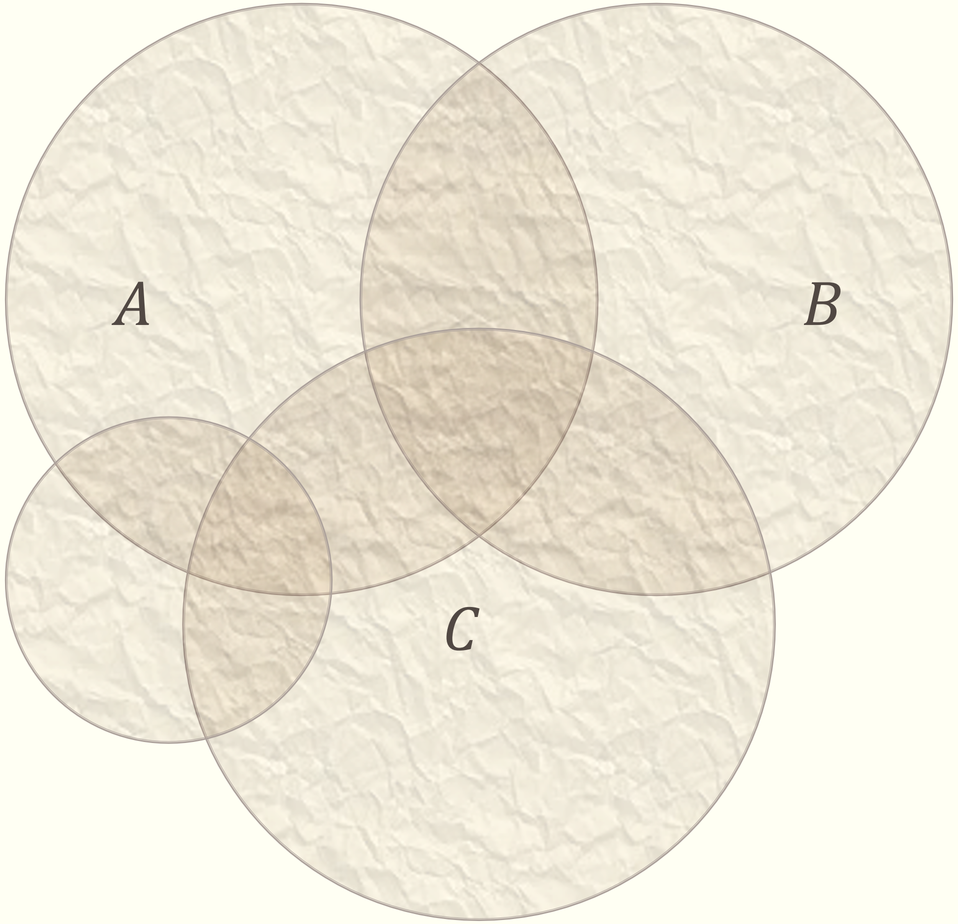
Important Properties

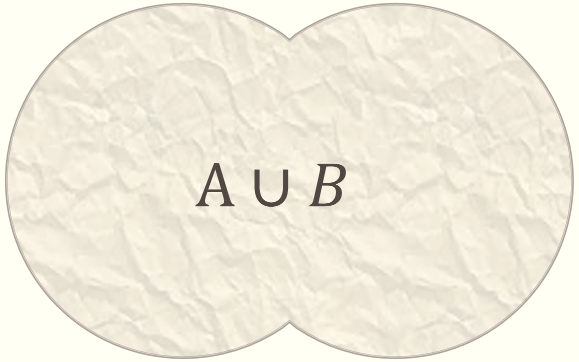
- If A and B are subsets of Ω, then

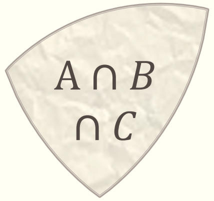
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- If A, B and C are subsets of Ω, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$-P(A \cap B) - P(B \cap C) - P(A \cap C)$$
$$+P(A \cap B \cap C).$$







We can generalize the formula after we learn permutation and combination.

!

Uniform Distribution

The uniform distribution on a sample space Ω containing n elements is the function m defined by

$$m(\omega) = \frac{1}{n},$$

for every $\omega \in \Omega$.

Examples:

- ?
- ?
- ?

Draw a poker card

1, 2, 3, …, 10, J, Q, K

He deals the cards as a meditation
And those he plays never suspect
He doesn't play for the money he wins
He don't play for respect
He deals the cards to find the answer
The sacred geometry of chance
The hidden law of a probable outcome
The numbers lead a dance

Draw a poker card
spade, club, diamond, heart
I know that the **spades** are the swords of a soldier I know that the **clubs** are weapons of war I know that **diamonds** mean money for this art But that's not the shape of my **heart** He may play the **jack** of diamonds He may lay the **queen** of spades He may conceal a **king** in his hand While the memory of it fades

XC 2020

Draw a poker card

$(10+3)\times 4$

$$m(\omega) = \cdots$$

Draw a poker card: French suits

spade, diamond, club, heart

$$m(\omega) = \cdots$$

Draw a poker card: ranks

1, 2, 3, ..., 10, J, Q, K

$$m(\omega) = \cdots$$

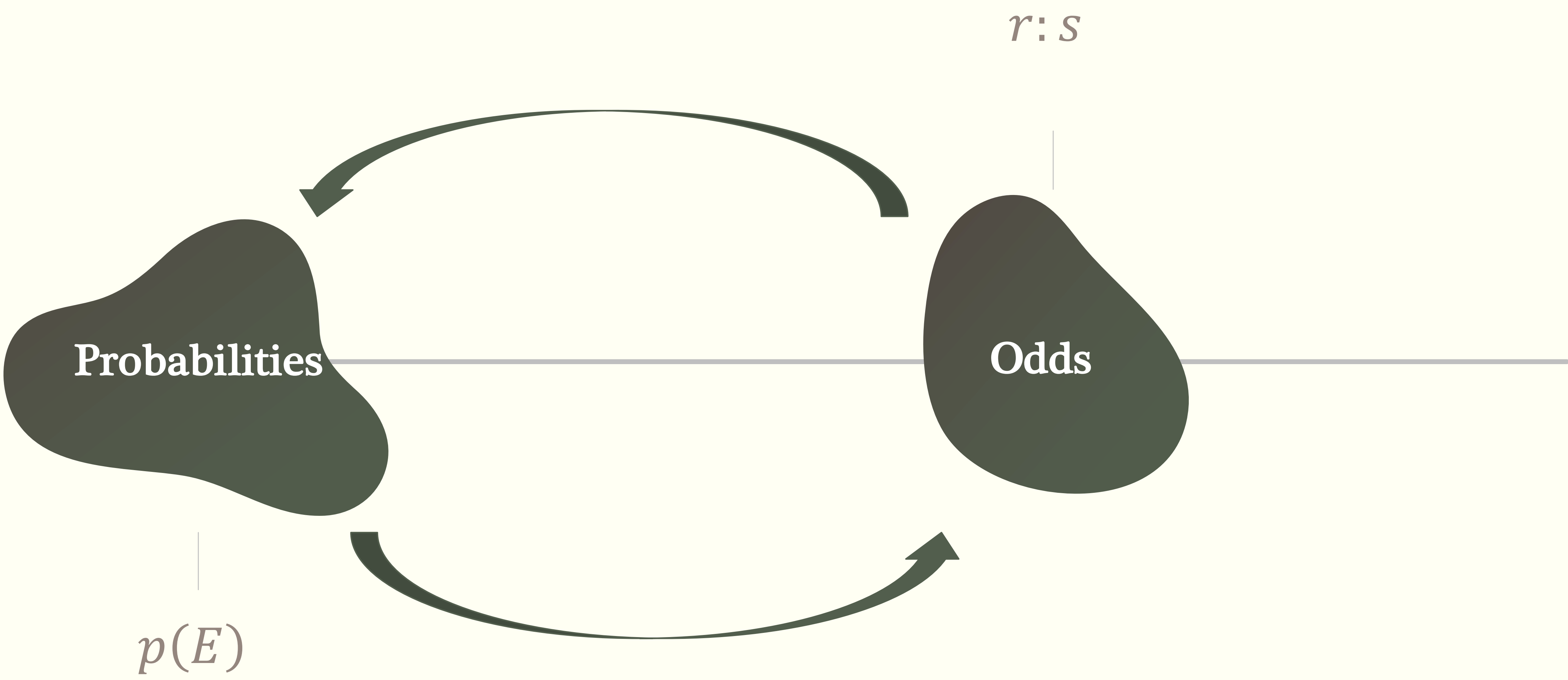
Draw a poker card
$(10+3)\times 4$
$m(\omega)=\frac{1}{52}$

Draw a poker card: French suits
spade, diamond, club, heart
$m(\omega)=\frac{1}{4}$

Draw a poker card: ranks
1, 2, 3, …, 10, J, Q, K
$m(\omega)=\frac{1}{13}$

XC 2020

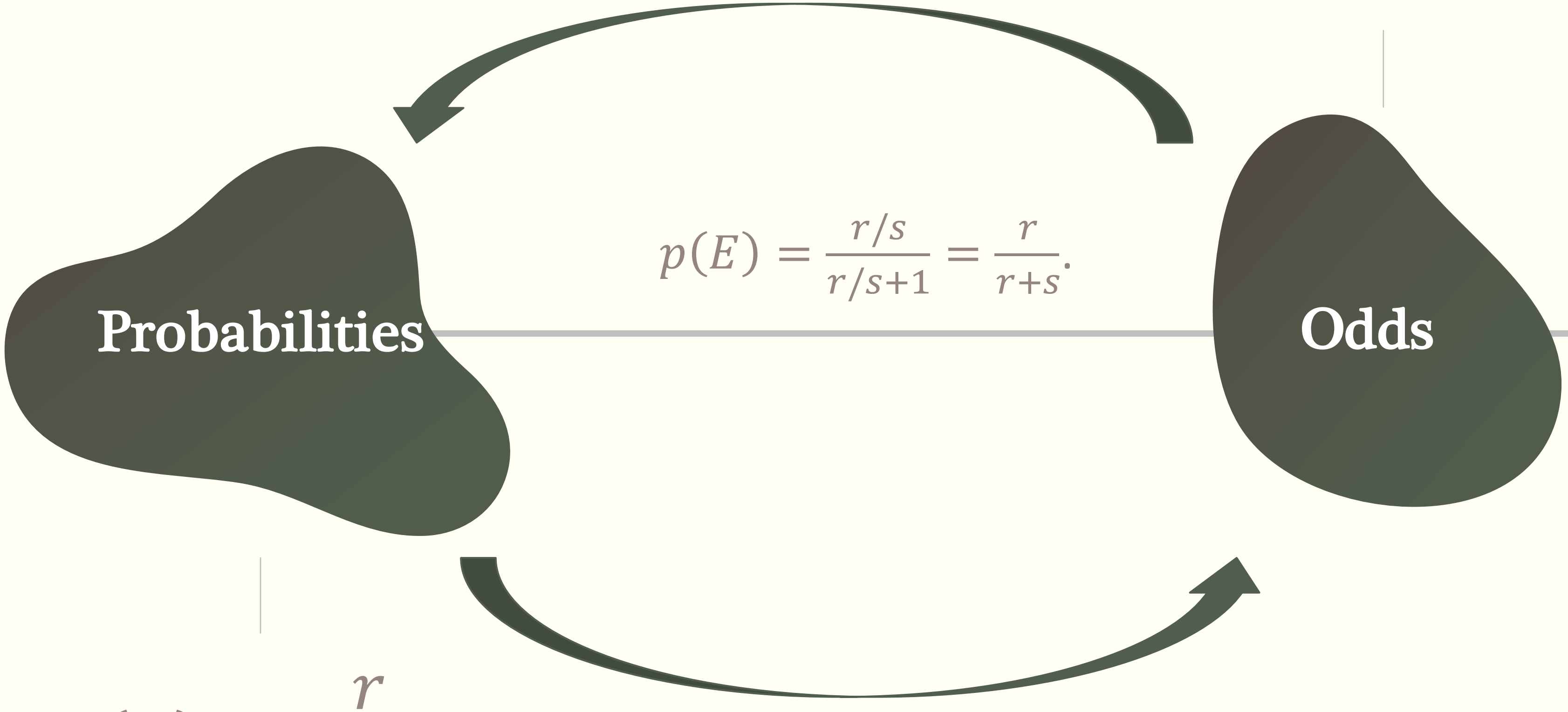
Determination of Probabilities



Determination of Probabilities

If r and s are given, then p can be found by using the equation $p = r/(r + s)$.

$r : s$



$$p(E) = \frac{r/s}{r/s+1} = \frac{r}{r+s}.$$

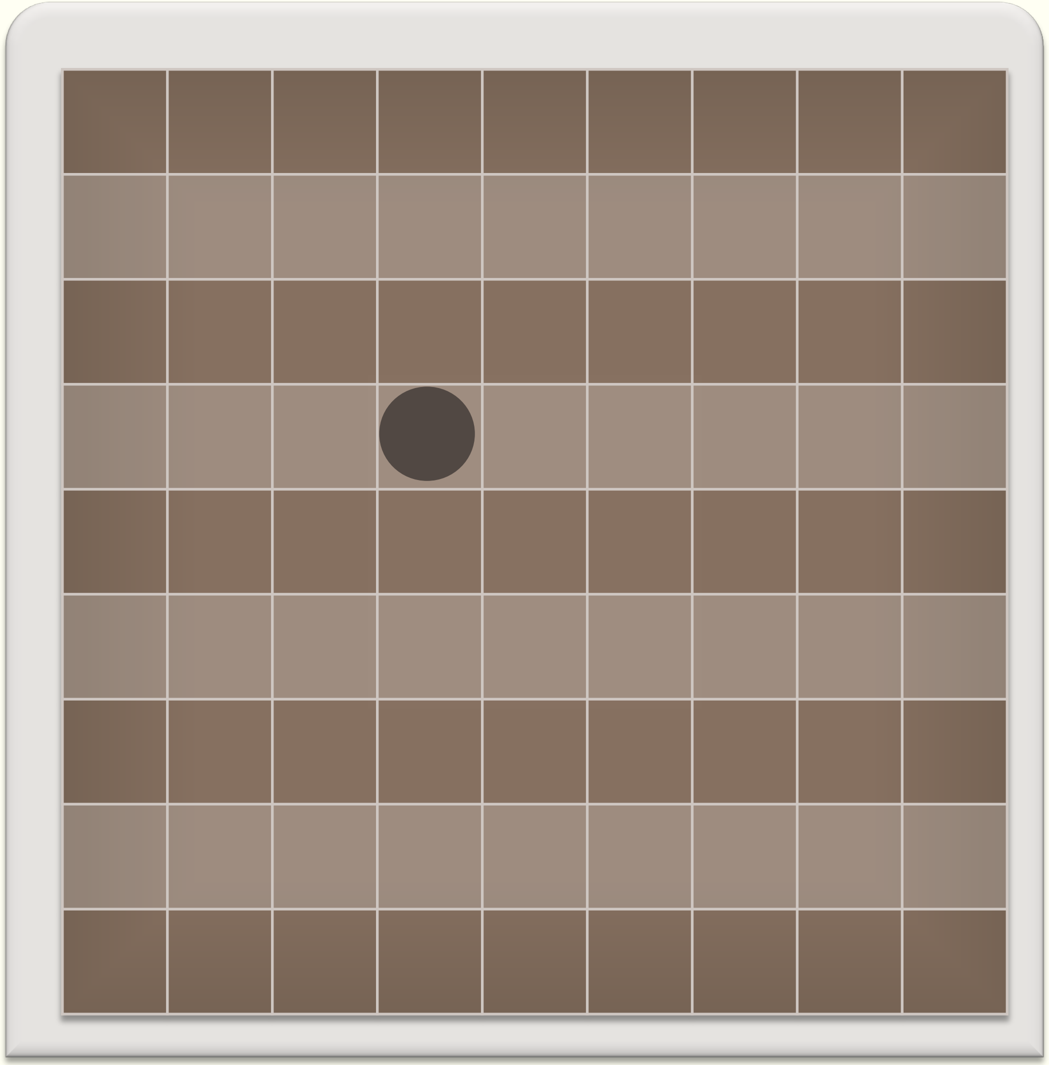
$$p(E) = \frac{r}{r+s}$$

If $P(E) = p$, the odds in favor of the event E occurring are $r : s$ (r to s) where $r/s = p/(1 - p)$.

Infinite Sample Space

A sample space is **countably infinite** if the elements can be counted, i.e., can be put in one-to-one correspondence with the positive integers, and uncountably infinite otherwise (which requires the concepts of continuous probability densities).

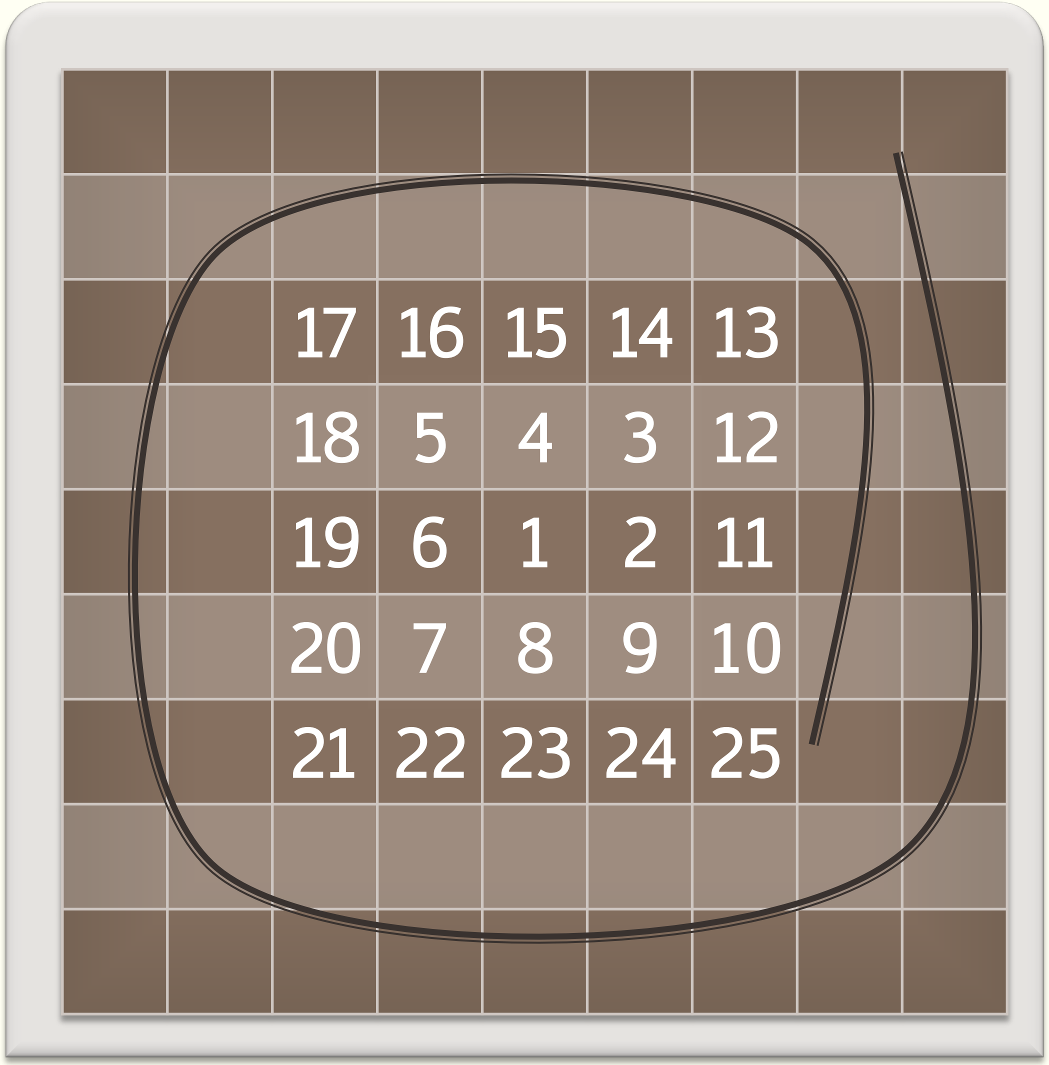
Choose a square on an infinite chessboard	
sample space	
elementary outcome	
probability distribution function	



Infinite Sample Space

A sample space is **countably infinite** if the elements can be counted, i.e., can be put in one-to-one correspondence with the positive integers, and uncountably infinite otherwise (which requires the concepts of continuous probability densities).

Choose a square on an infinite chessboard	
sample space	$\Omega = \{1, 2, 3, 4, 5, 6, \cdots\}$
elementary outcome	$\omega = 1, 2, 3, 4, 5, 6, \cdots$
probability distribution function	$m(\omega) = \cdots$



Infinite Sample Space

If $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$ is a countably infinite sample space, then a distribution function can be defined as in the case of a finite sample space, but now the infinite sum must be convergent (and thus cannot be uniform).

A distribution function for X is a real-valued function m whose domain is Ω and which satisfies:

- $m(\omega) \geq 0$, for all $\omega \in \Omega$, and
- $\sum_{\omega \in \Omega} m(\omega) = 1$.

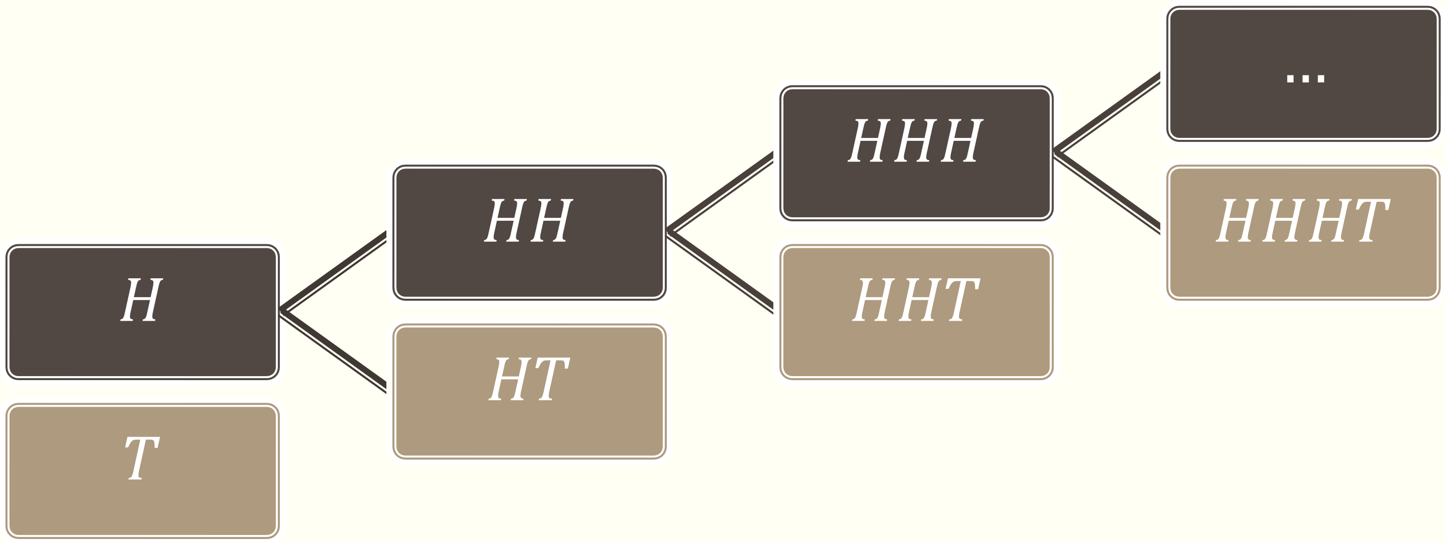
Choose a square on an infinite chessboard	
sample space	$\Omega = \{1, 2, 3, 4, 5, 6, \cdots\}$
elementary outcome	$\omega = 1, 2, 3, 4, 5, 6, \cdots$
probability distribution function	$m(\omega) = 0$

Infinite Discrete Sample Space

First Tail

- The experiment is to repeatedly toss a coin until first tail shows up.
- Possible outcomes are sequences of H that, if finite, end with a single T, and an infinite sequence of H:

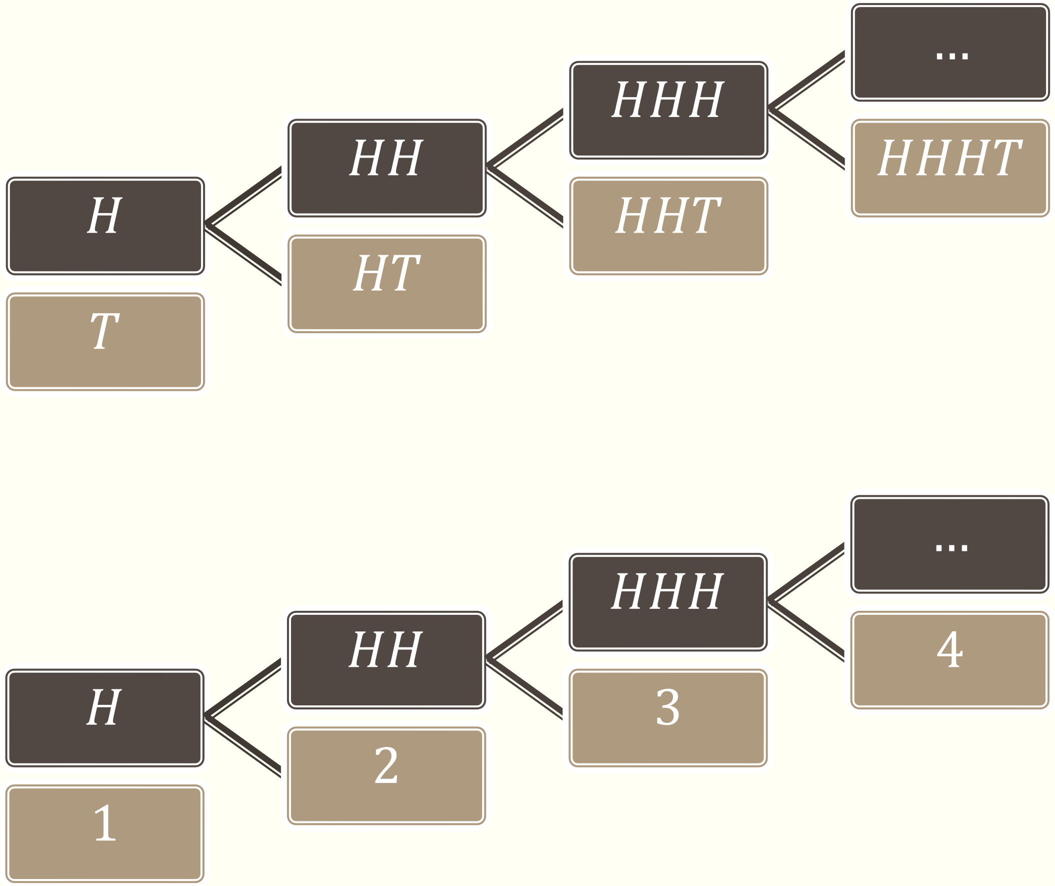
$$\Omega = \{T, HT, HHT, HHHT, HHHHT, \ldots\}$$



Infinite Discrete Sample Space

First Tail

- The experiment is to repeatedly toss a coin until first tail shows up.
- Possible outcomes are sequences of H that, if finite, end with a single T, and an infinite sequence of H:
- $\Omega = \{T, HT, HHT, HHHT, HHHHT, \dots\}$
- One random variable is defined most naturally as the length of an outcome.
- It draws values from the set of whole numbers augmented by the symbol of infinity:
- $\{1, 2, 3, 4, 5, \dots, \infty\}$



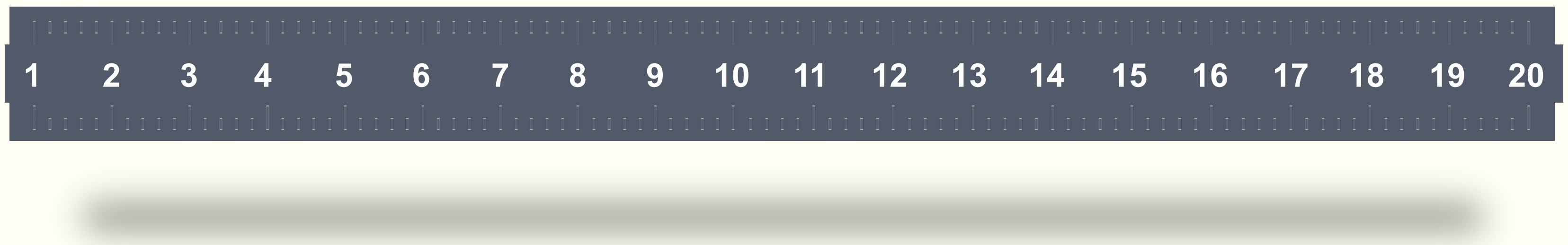
Continuous Sample Space

arrival time

5:45 pm – 6:00 pm

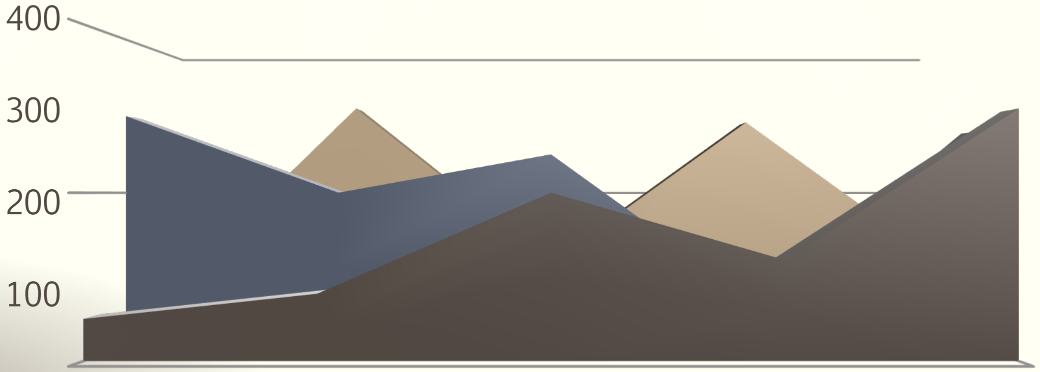
speed

160 mph – 200 mph



length

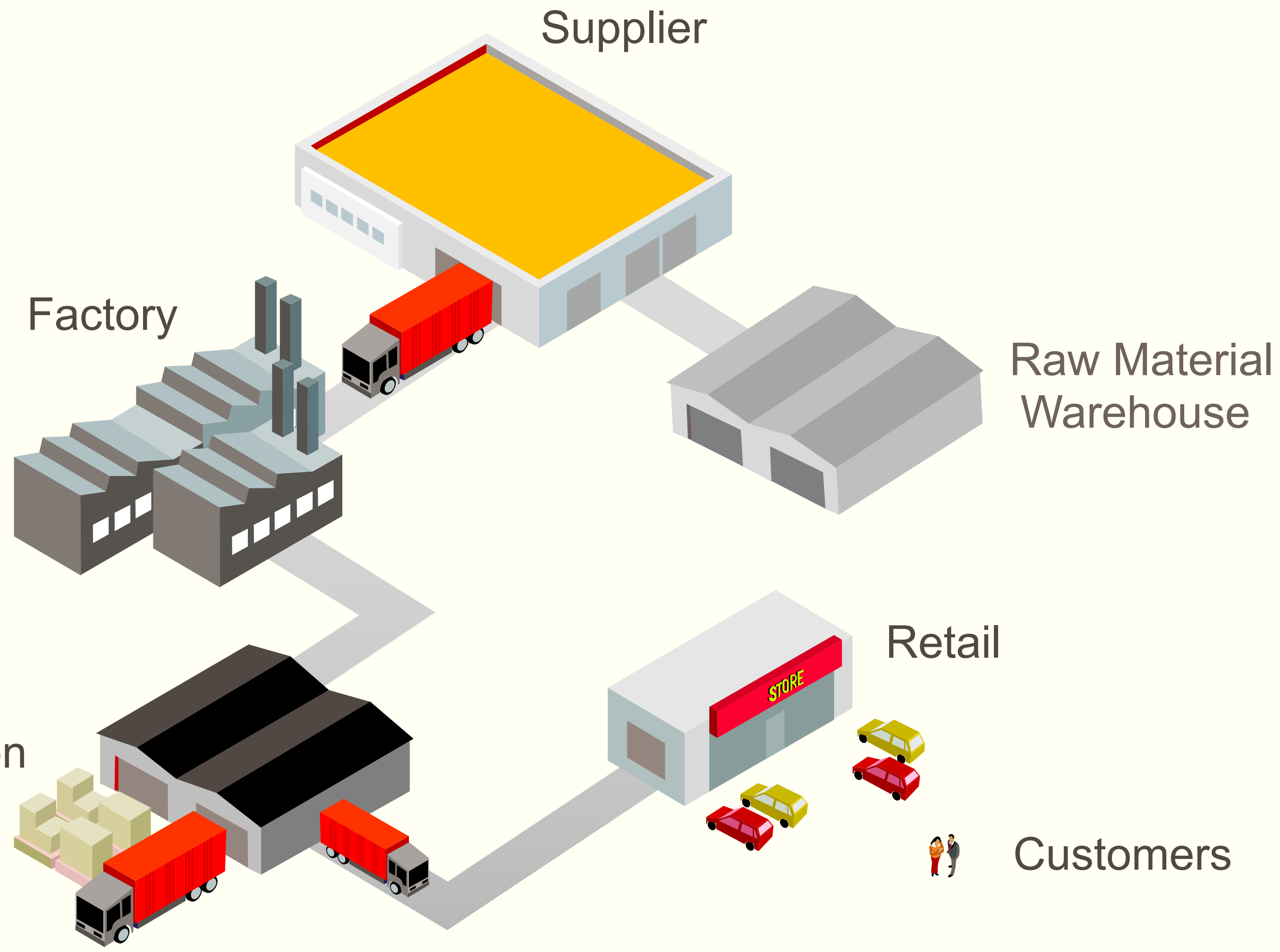
time



area

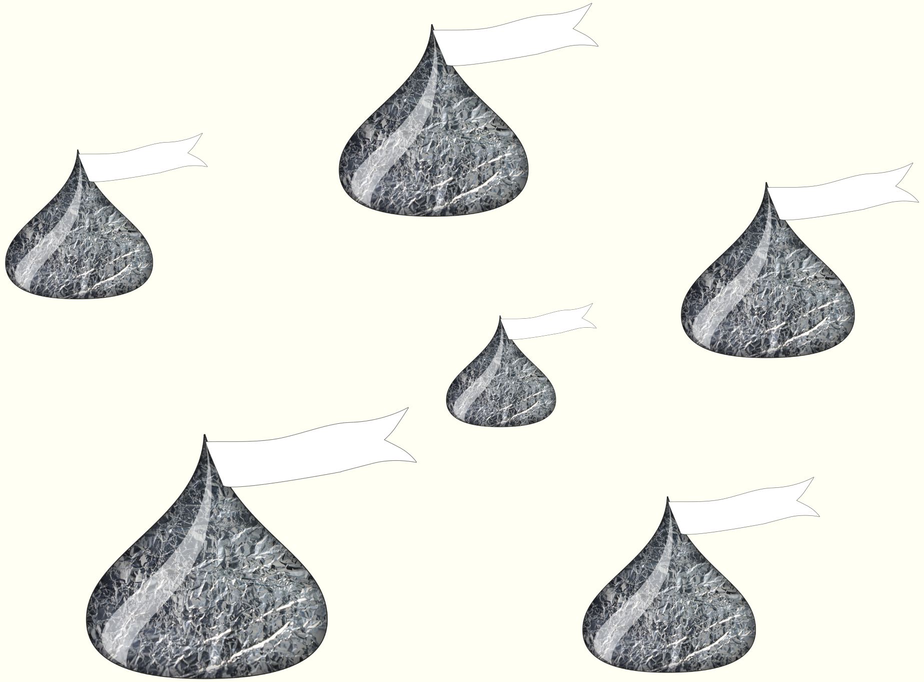
velocity (speed)

Processes that Operate Efficiently and Produce Items of the Highest Quality



How Much Does a Hershey Kiss Weight?

- A single standard Hershey's Kiss weighs 0.16 ounces.

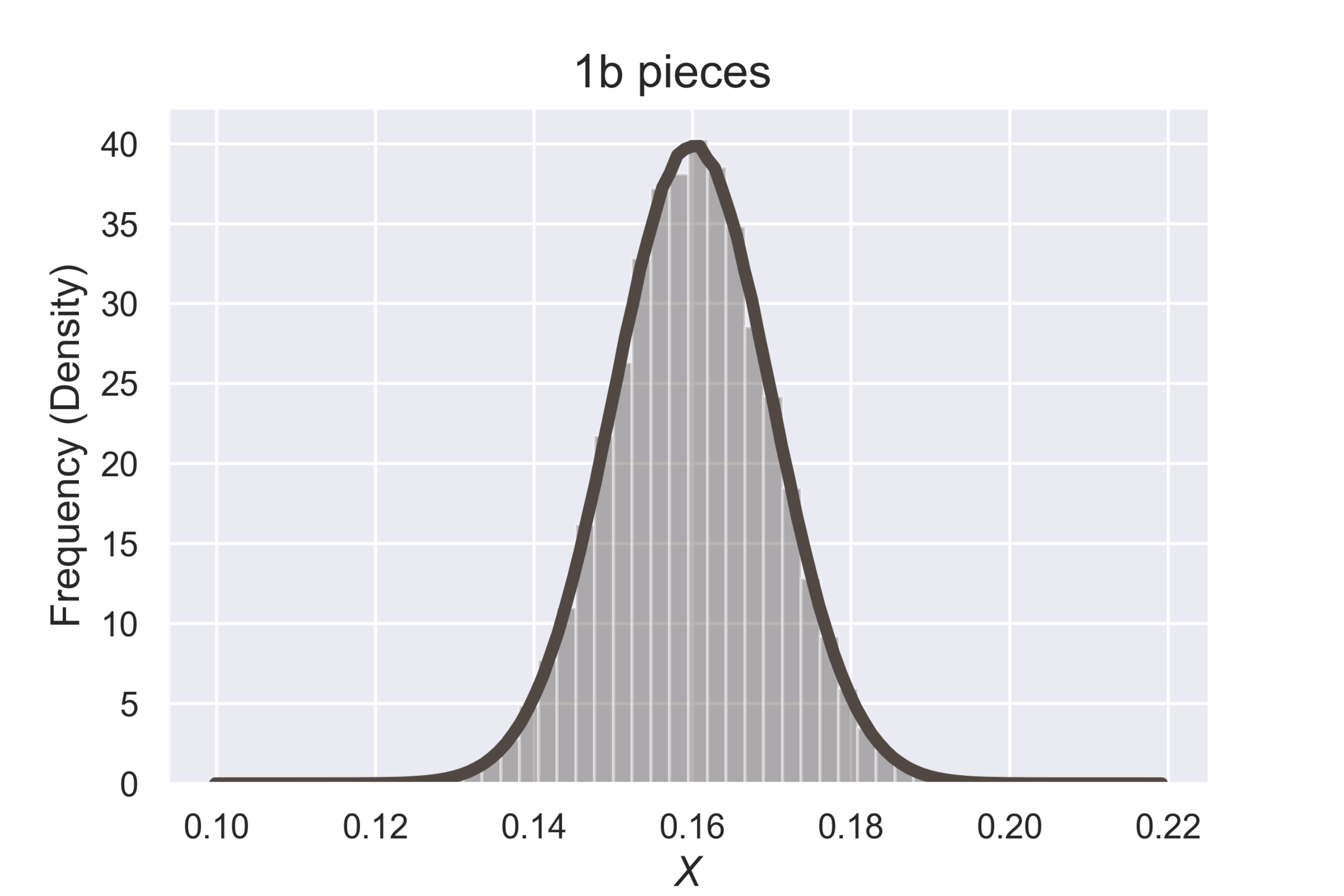


How Much Does a Hershey Kiss Weight?

- A single standard Hershey's Kiss weighs 0.16 ounces.

0.1584	0.1577	0.1819	0.1581	0.1438	0.1385	0.1673	0.1611
0.165	0.1452	0.1482	0.1568	0.1603	0.1478	0.1591	0.1519
0.1649	0.1672	0.153	0.1504	0.1587	0.1485	0.1538	0.1498
0.1656	0.1692	0.1477	0.157	0.1574	0.1699	0.1589	0.1487

Normal Density Distribution (Gaussian Distribution)



The normal density function with parameters μ and σ
expectation: μ, standard deviation: σ

- Three-sigma limits is a statistical calculation that refers to data within three standard deviations from a mean.
- In business applications, three-sigma refers to processes that operate efficiently and produce items of the highest quality.
- Three-sigma limits are used to set the upper and lower control limits in statistical quality control charts.
- Control charts are used to establish limits for a manufacturing or business process that is in a state of statistical control.