MATH 20: PROBABILITY

Continuous **Conditional** Probability

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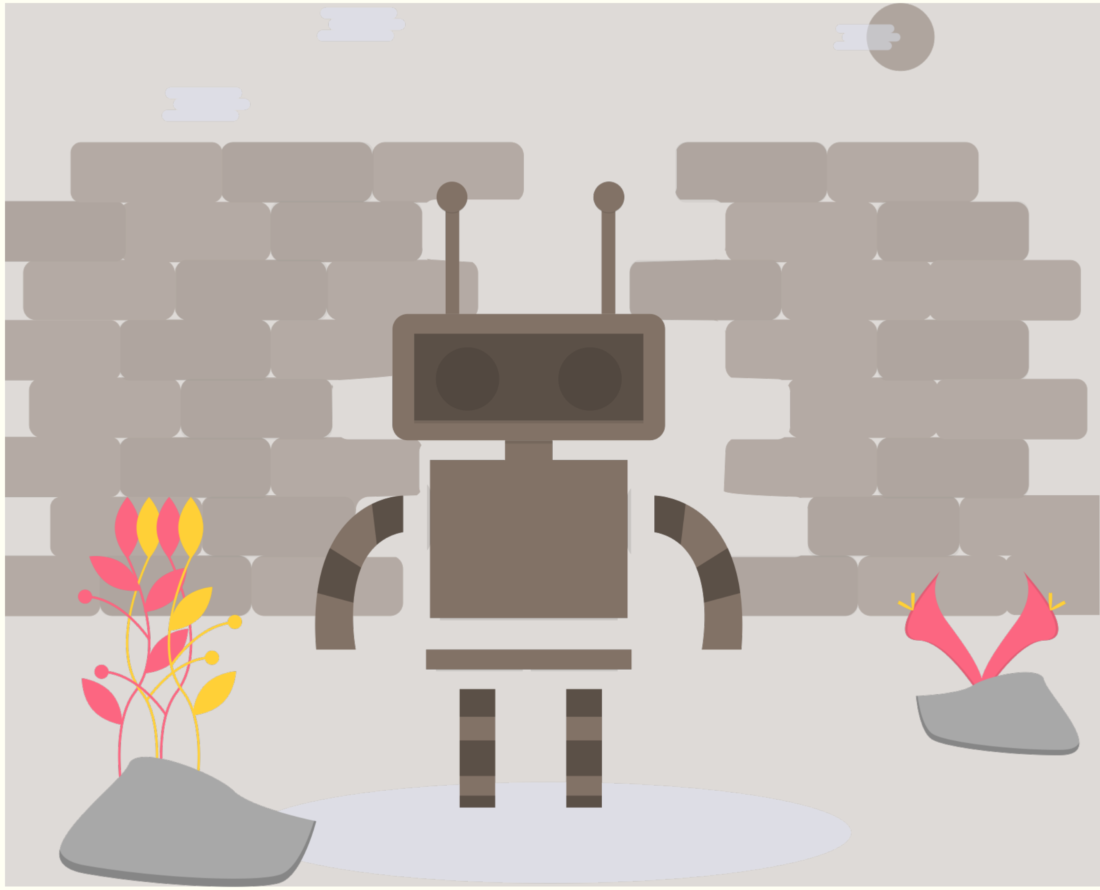
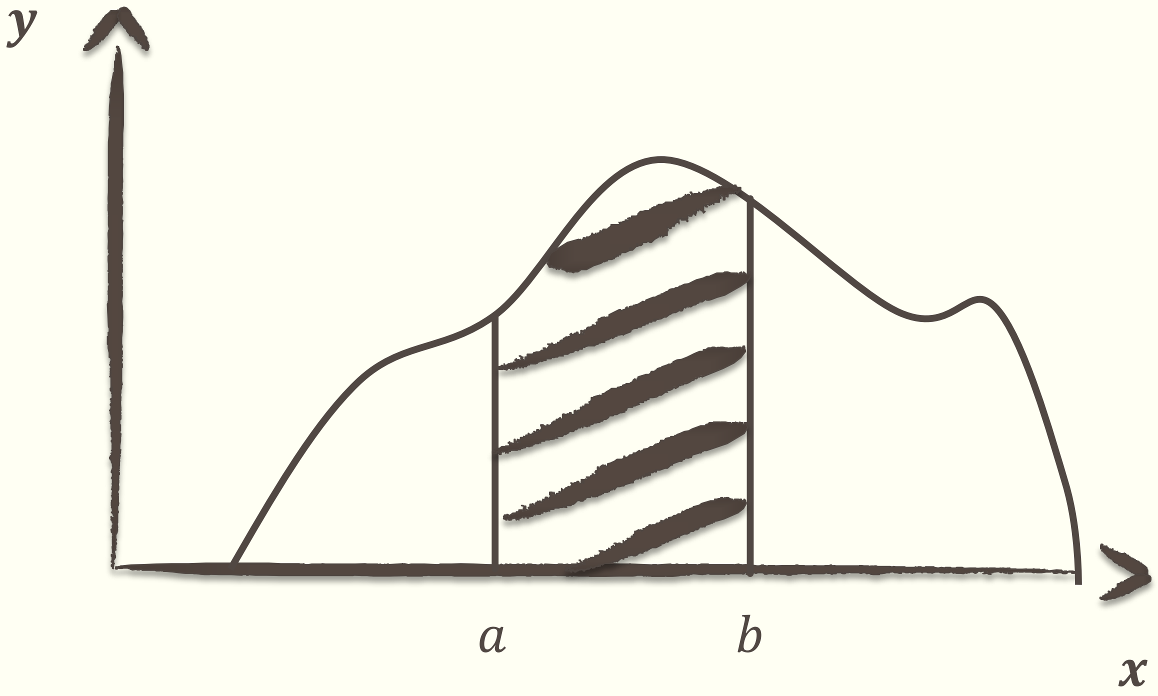


Density Functions of Continuous Random Variable

- Assume X is continuous random variable with density function $f(x)$, and if E is an event with positive probability, we define conditional density function (which is normalized to have integral 1 over E):

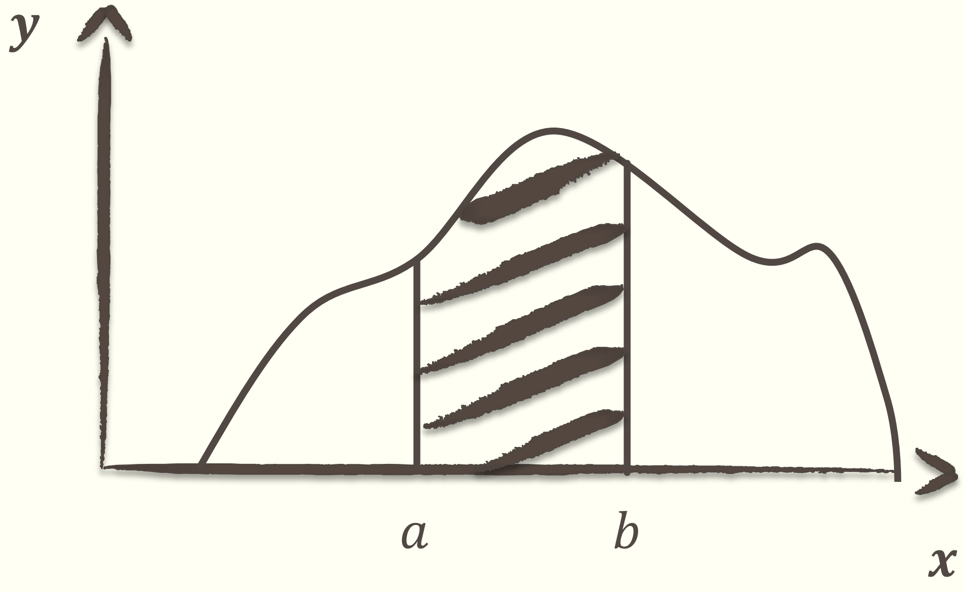
$$f(x|E) = \begin{cases} f(x)/P(E), & x \in E \\ 0, & x \notin E \end{cases}$$

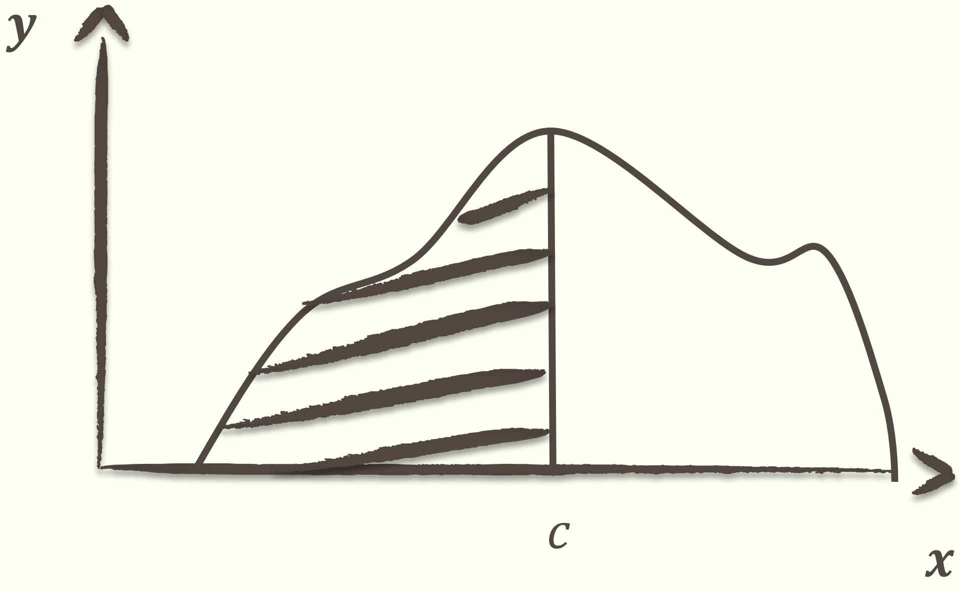


Density Functions of Continuous Random Variable

- For any event F, the conditional probability of F given E is given by

$$P(F|E) = \int_F f(x|E)dx = \int_{E\cap F} \frac{f(x)}{P(E)} dx = \frac{P(E\cap F)}{P(E)}.$$





Density Functions of Continuous Random Variable

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$$P(F|E) = \int_F f(x|E)dx = \int_{E\cap F} \frac{f(x)}{P(E)} dx = \frac{P(E\cap F)}{P(E)}.$$

- Event E and F are independent

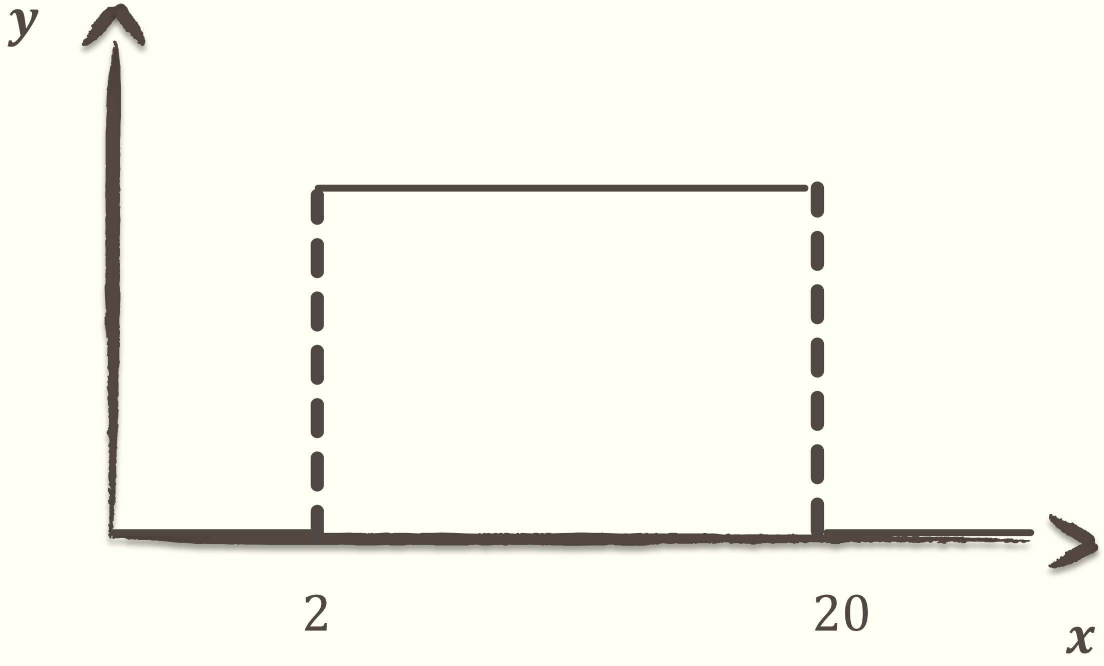
$$\Leftrightarrow$$

$$P(E \cap F) = P(E)P(F)$$

$$\Leftrightarrow$$

$$P(F|E) = P(F) \text{ and } P(E|F) = P(E).$$

Uniform Distribution



$$f(x) = \begin{cases} \frac{1}{18}, & 2 \le x \le 20 \\ 0, & x < 2 \text{ or } x > 20 \end{cases}$$

Event E: $X > 10$

$$P(X > 10) = \frac{5}{9}$$

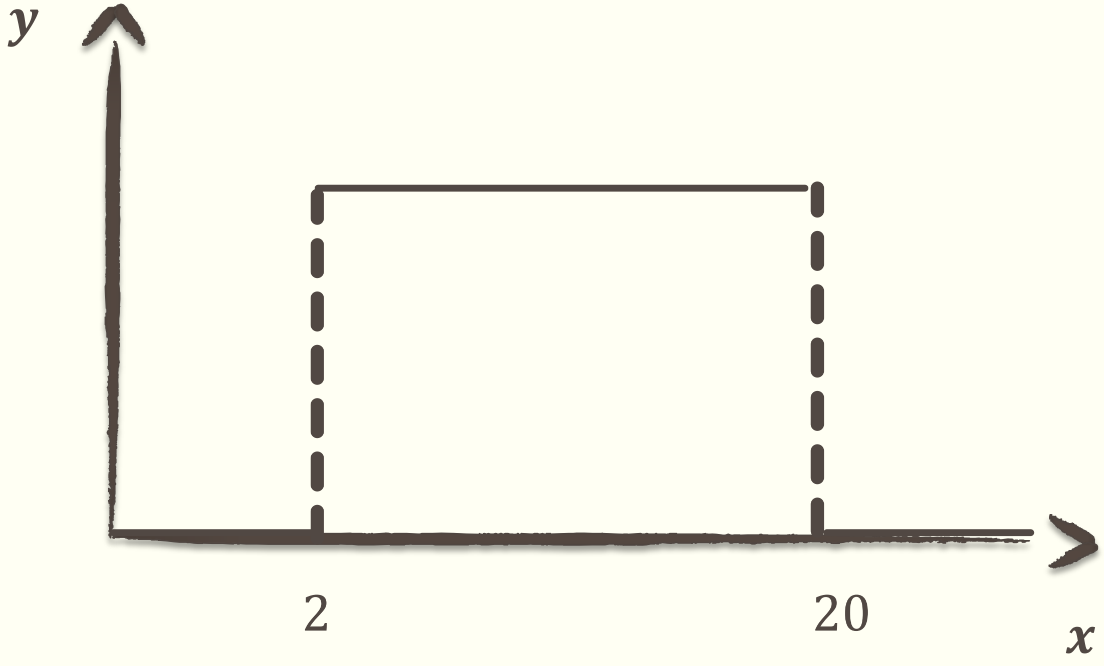
Event F: $X > 12$

$$P(X > 12) = \frac{4}{9}$$

Event $F|E$: $X > 12|X > 10$

$$P(\boldsymbol{X} > \mathbf{12}|\boldsymbol{X} > \mathbf{10}) = \cdots$$

Uniform Distribution



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$$P(X > 10) = \frac{5}{9}$$

Event F: $X > 12$

$$P(X > 12) = \frac{4}{9}$$

Event $F|E$: $X > 12|X > 10$

$$P(\boldsymbol{X} > \mathbf{12}|\boldsymbol{X} > \mathbf{10}) = \frac{P(X > 12 \cap X > 10)}{P(X > 10)}$$

$$= \frac{P(X > 12)}{P(X > 10)} = \frac{4}{5}$$

Exponential Distribution

- The exponential distribution is the probability distribution of the **time** between events in a Poisson point process. That is, a process in which events occur continuously and independently **at a constant average rate $\boldsymbol{\lambda}$**.

- The density function of an exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

- The cumulative distribution function of an exponential distribution is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Exponential Distribution

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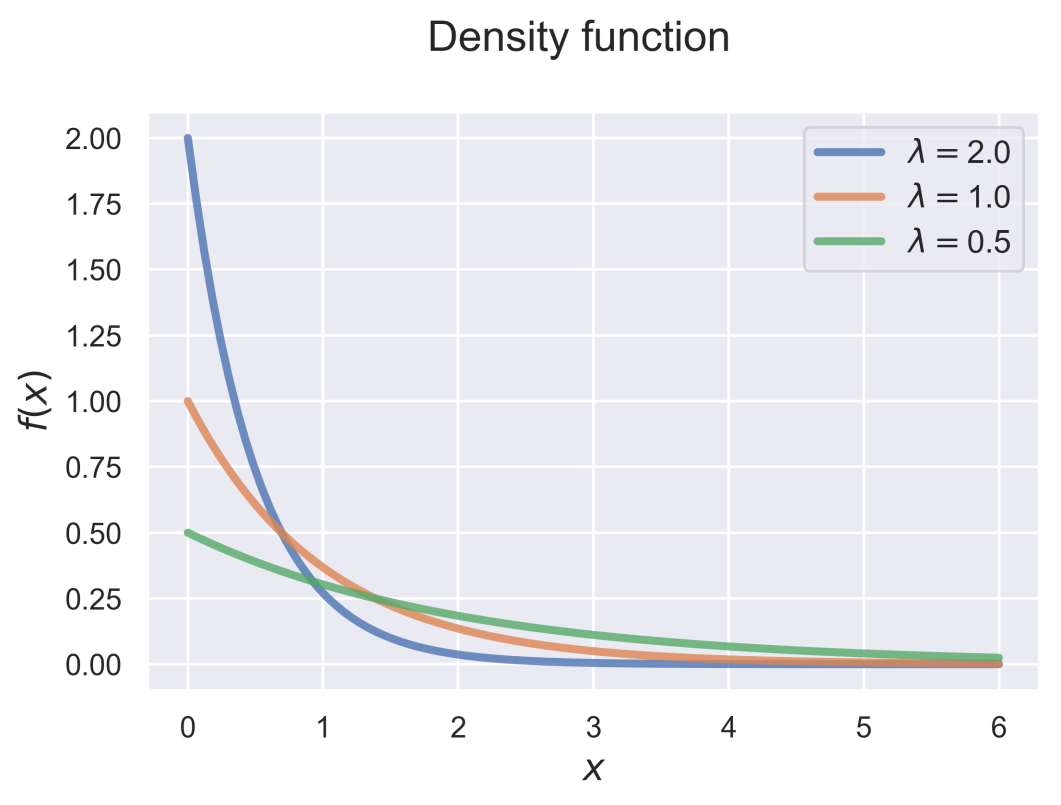
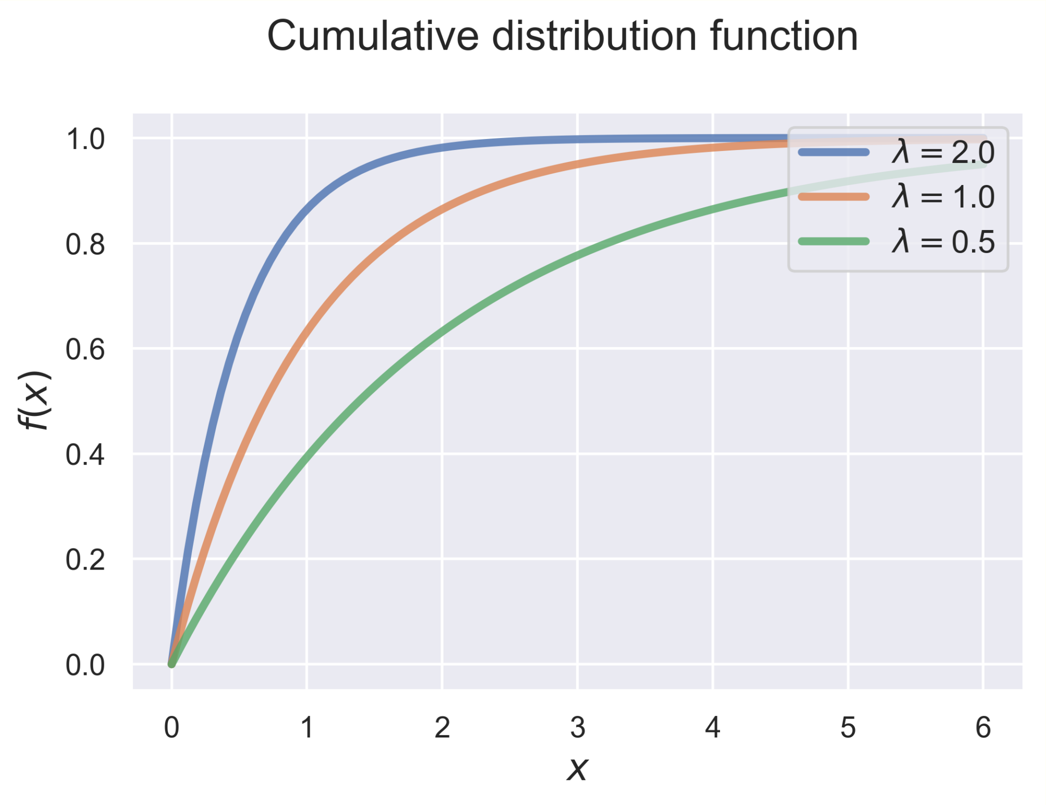
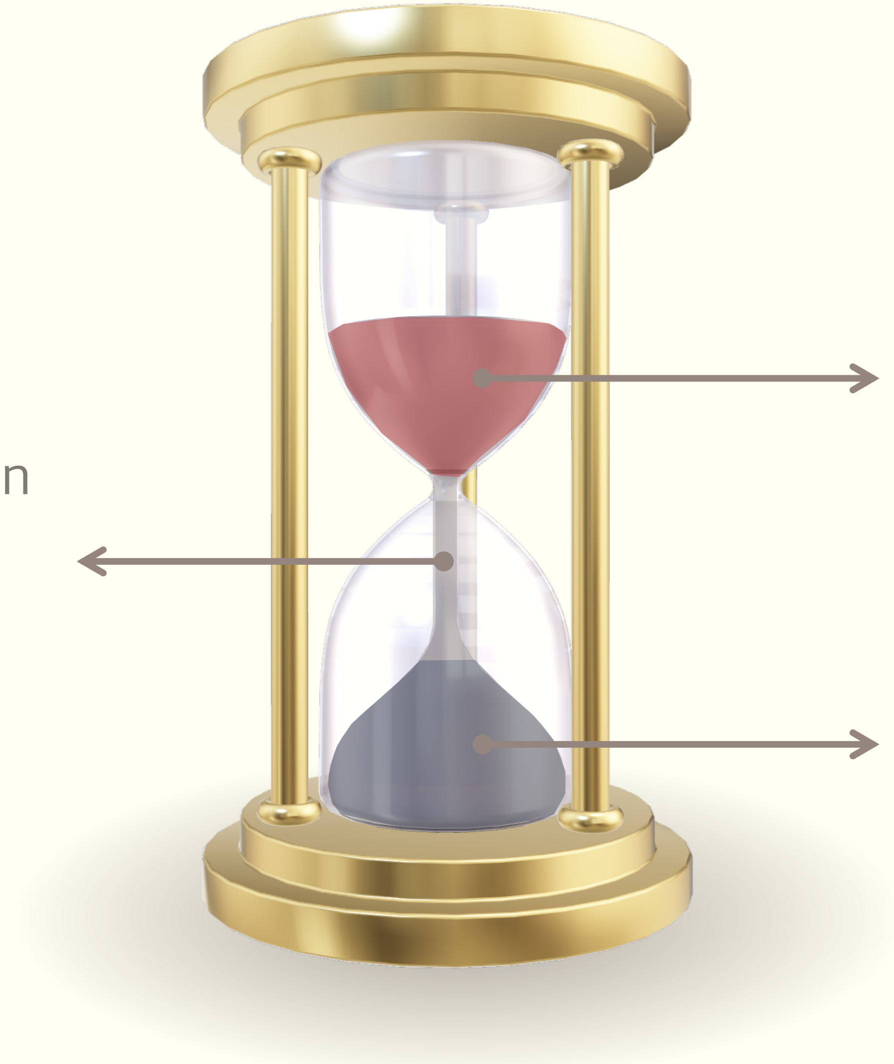




Exponential Distribution



Density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Time

The exponential distribution is often concerned with the amount of time until some specific event occurs.

Cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

The Life Span of a Lightbulb

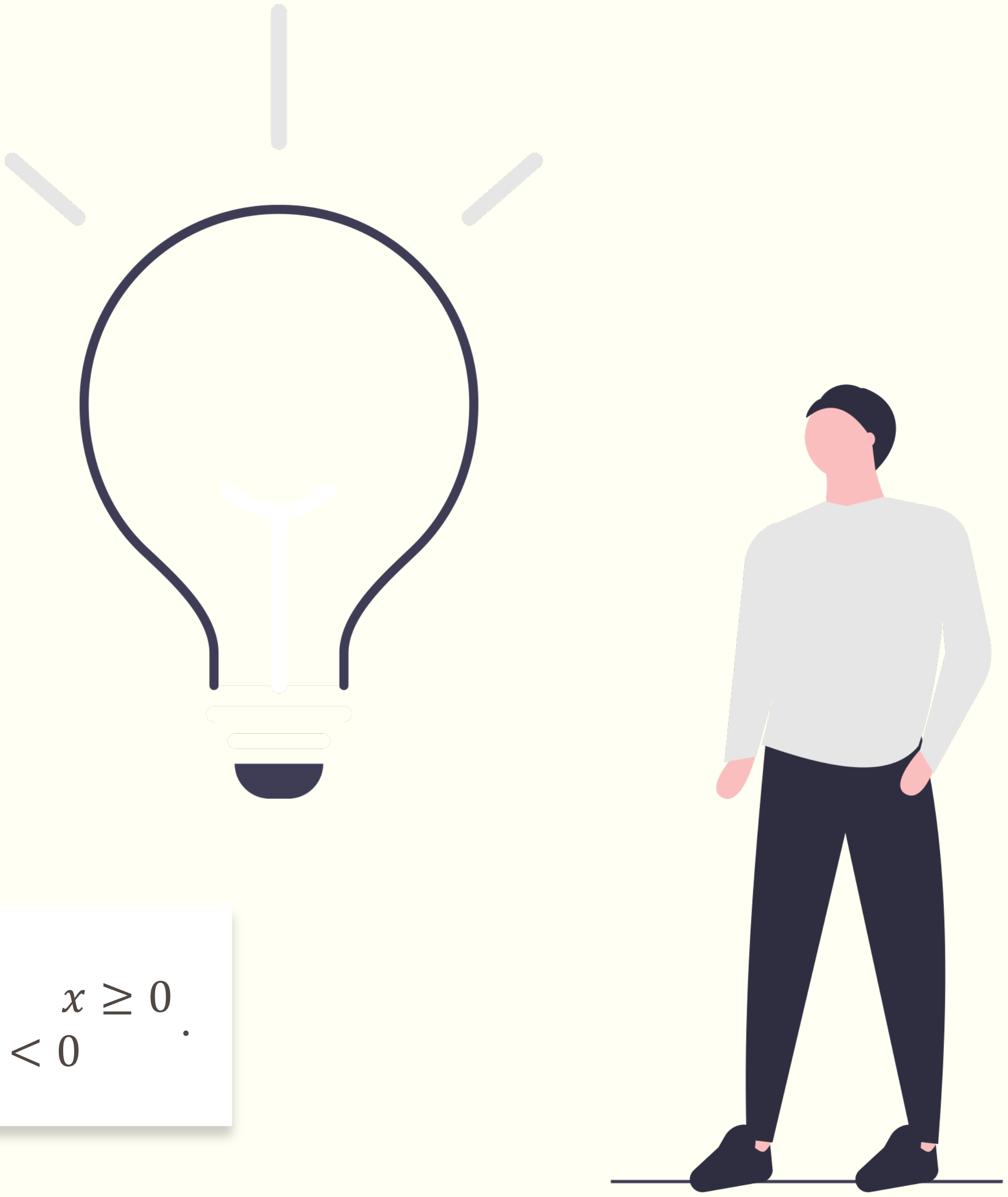
- Assume the life span of a lightbulb is a random variable t with an exponential density function. The average lifetime is 30 months.

average: μ
$\mu = 30$

rate: λ
$\lambda = \frac{1}{\mu}$

$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$



The Life Span of a Lightbulb

- **If the lightbulb is already lit for 15 months**, how long it will continue to last until burning out?

$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

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$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Event E: $X > 15$

$$P(X > 15) = 1 - P(X \leq 15) = e^{-15/30}$$
$$= e^{-1/2}$$

Event F: $X > 15 + s$

$$P(X > 15 + s) = 1 - P(X \leq 15 + s)$$
$$= e^{-(15+s)/30} = e^{-1/2} e^{-s/30}$$

The Life Span of a Lightbulb

- If the lightbulb is already lit for 15 months, how long it will continue to last until burning out?

Event E: $X > 15$

$$P(X > 15) = 1 - P(X \le 15) = e^{-15/30}$$
$$= e^{-1/2}$$

Event F: $X > 15 + s$

$$P(X > 15 + s) = 1 - P(X \le 15 + s)$$
$$= e^{-(15+s)/30} = e^{-1/2}e^{-s/30}$$

Event $F|E$: $X > 15 + s|X > 15$

$$P(X > 15 + s|X > 15) = \frac{P(X > 15 + s \cap X > 15)}{P(X > 15)} = \frac{P(X > 15 + s)}{P(X > 15)} = \frac{e^{-1/2}e^{-s/30}}{e^{-1/2}} = \boxed{e^{-s/30}}$$

The Life Span of a Lightbulb

$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Event $\boldsymbol{F|E{:}X > 15 + s|X > 15}$

$$P(X > 15 + s|X > 15) = \frac{P(X > 15 + s \cap X > 15)}{P(X > 15)} = \frac{P(X > 15 + s)}{P(X > 15)} = \frac{e^{-1/2} e^{-s/30}}{e^{-1/2}} = e^{-s/30}$$

Event $\boldsymbol{H}$: $\boldsymbol{X > s}$

$$P(X > s) = 1 - P(X \leq s) = e^{-s/30}$$

The Life Span of a Lightbulb

Event $F|E{:}X > 15 + s|X > 15$

$$P(X > 15 + s|X > 15) = \frac{P(X > 15 + s \cap X > 15)}{P(X > 15)} = \frac{P(X > 15 + s)}{P(X > 15)} = \frac{e^{-1/2}e^{-s/30}}{e^{-1/2}} = e^{-s/30}$$

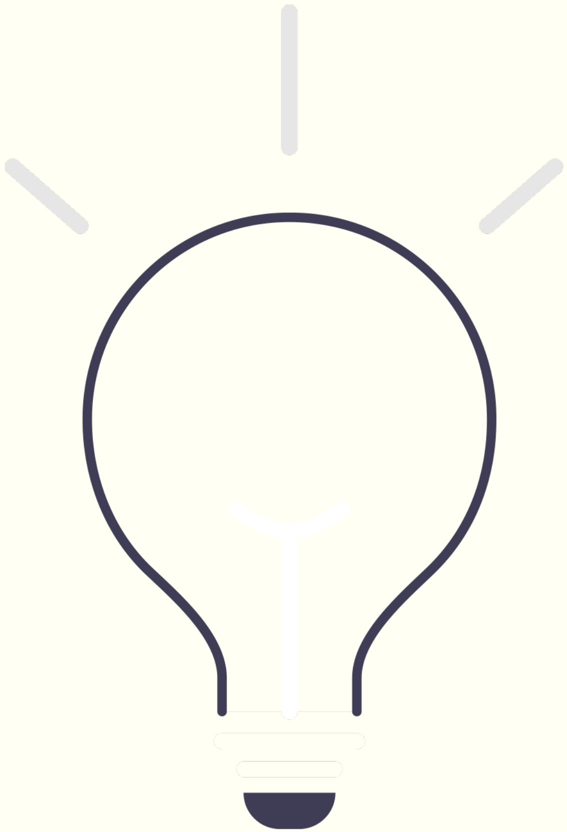
Event H: $X > s$

$$P(X > s) = 1 - P(X \leq s) = e^{-s/30}$$

$$P(X > 15 + s|X > 15) = P(X > s)$$

=

The Life Span of a Lightbulb



- Assume the life span of a lightbulb is a random variable t with an exponential density function.
- If the lightbulb is already lit for time r, what is the probability that it will not burn out for further time s?

average: μ	rate: λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$P(X > r + s | X > r) = \cdots$$

The Life Span of a Lightbulb

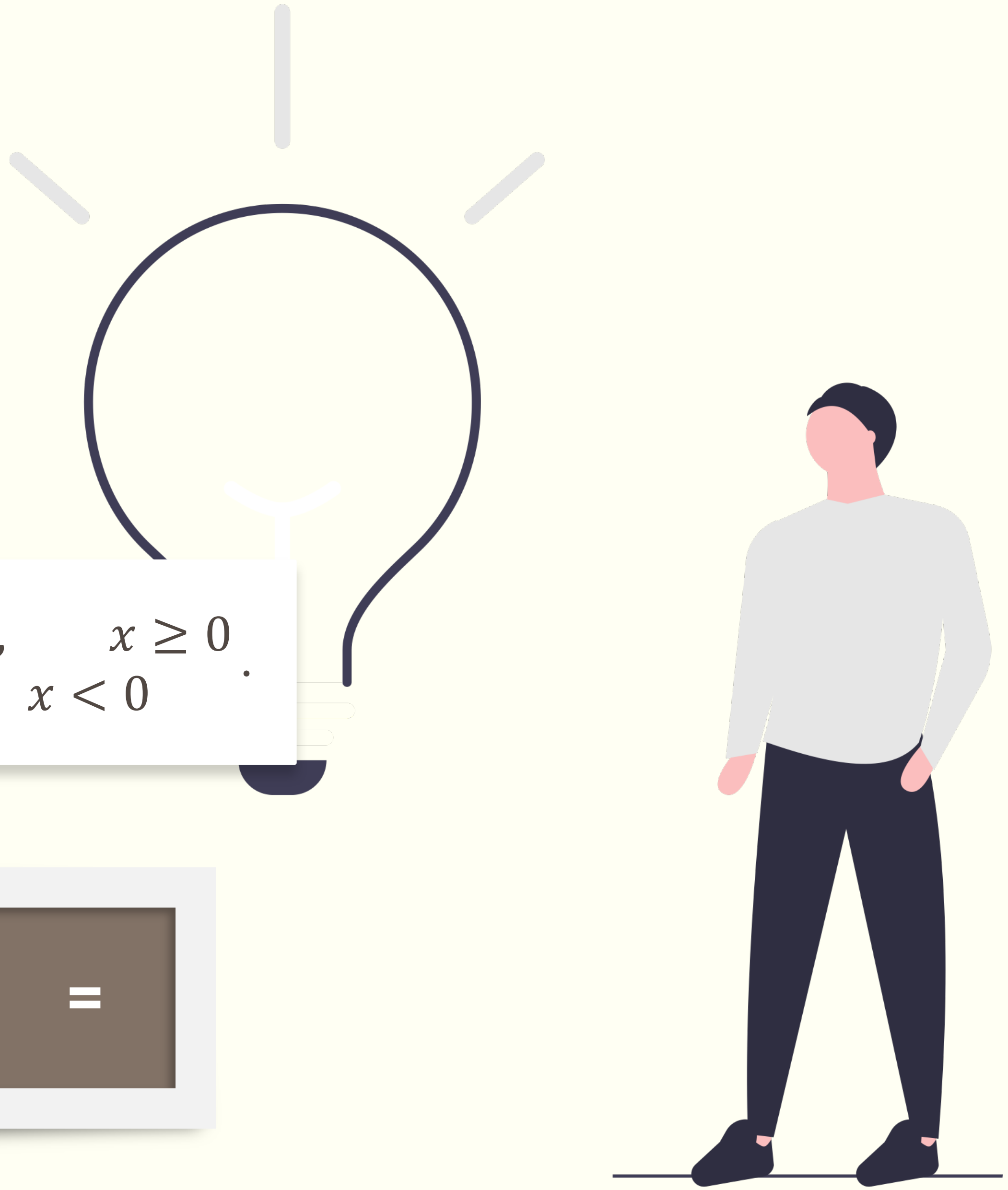
- Assume the life span of a lightbulb is a random variable t with an exponential density function.
- If the lightbulb is already lit for time r, what is the probability that it will not burn out for further time s?

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

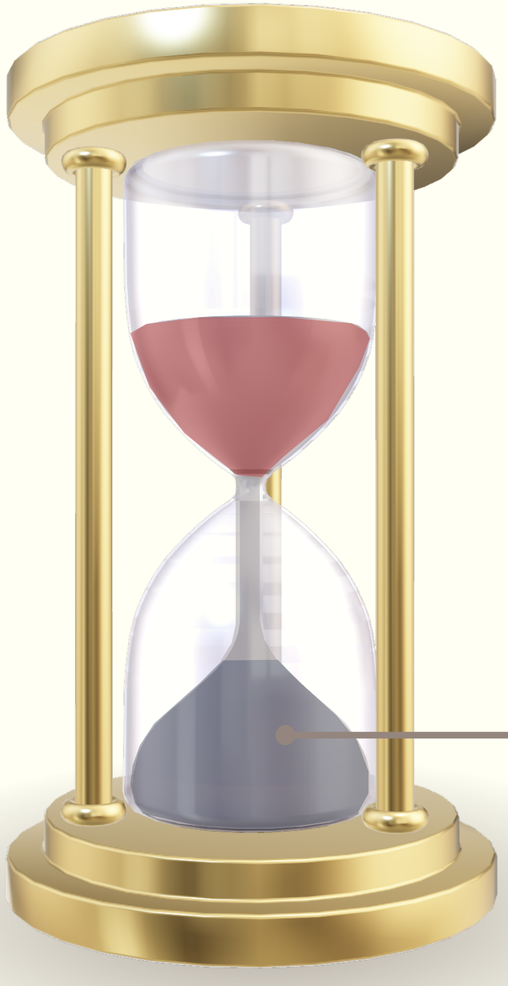
$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$P(X > r + s | X > r) = e^{-\lambda s} = P(X > s)$$

=



Exponential Distribution



Memoryless Property

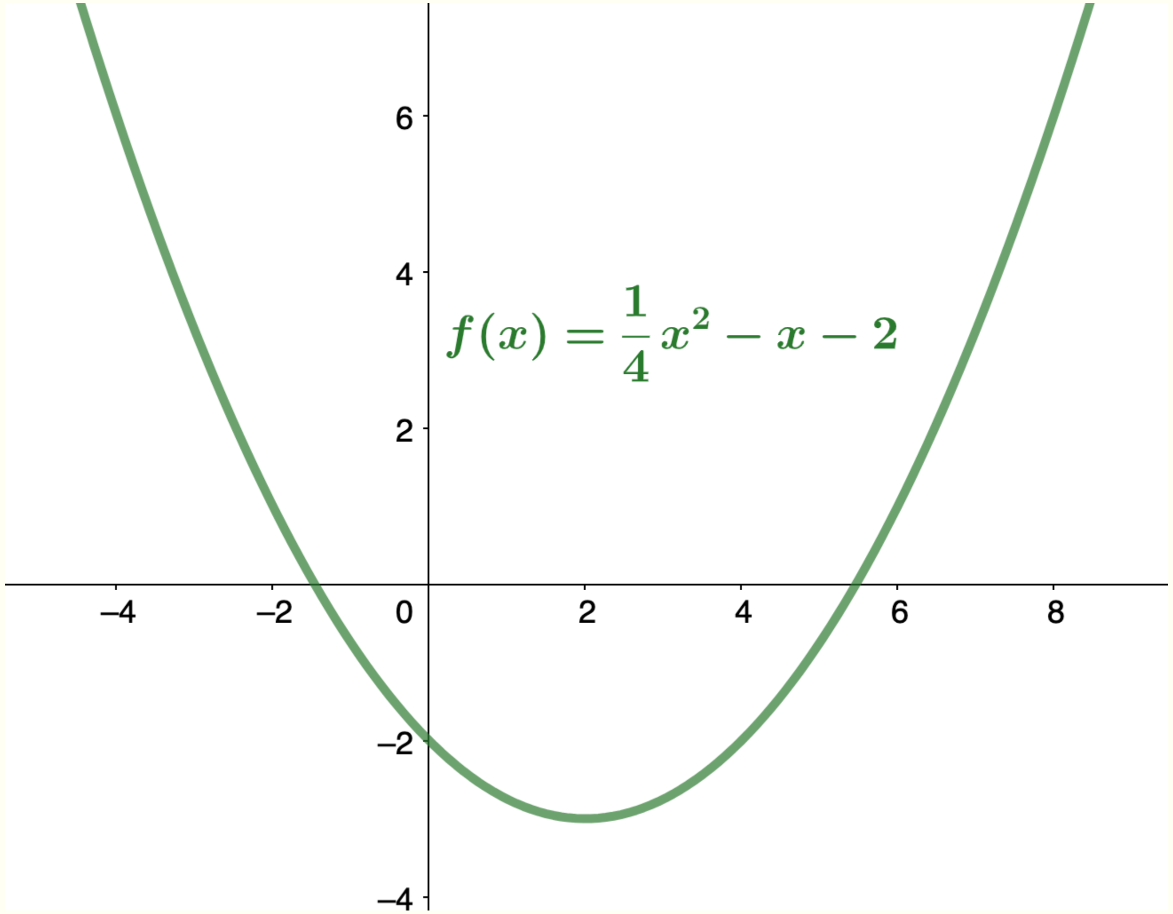
$$P(X > r + s | X > r) = P(X > s)$$

Time

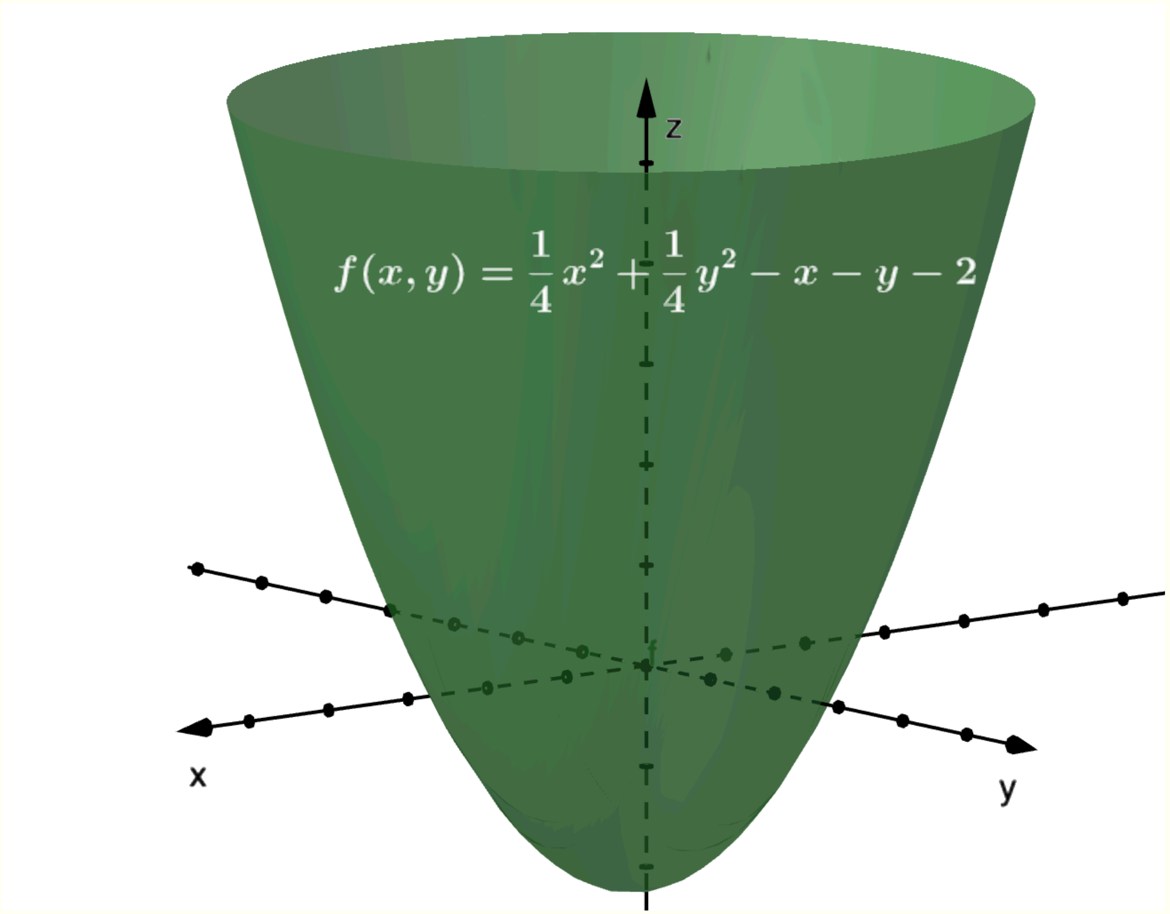
- The amount of time we have to wait for an occurrence does not depend on how long we have already waited.
- The memoryless property says that knowledge of what has occurred in the past has no effect on future probabilities.

JOINT DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS

For continuous random variables



$f(x) = \frac{1}{4}x^2 - x - 2$
6
4
2
–2
–4
–4
–2
0
2
4
6
8



$f(x,y) = \frac{1}{4}x^2 + \frac{1}{4}y^2 - x - y - 2$
z
x
y

Joint Density and Cumulative Distribution Functions

- Let $X_1, X_2, \cdots, X_n$ be continuous random variables associated with an experiment. And let $X = (X_1, X_2, \cdots, X_n)$.
- The joint cumulative distribution function of X is defined by

$$F(x_1, x_2, \cdots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n).$$

- The joint density function of X, $f(x_1, x_2, \cdots, x_n)$, satisfies the following equations:

$$F(x_1, x_2, \cdots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(x_1, x_2, \cdots, x_n)\, dt_n dt_{n-1} \cdots dt_1.$$

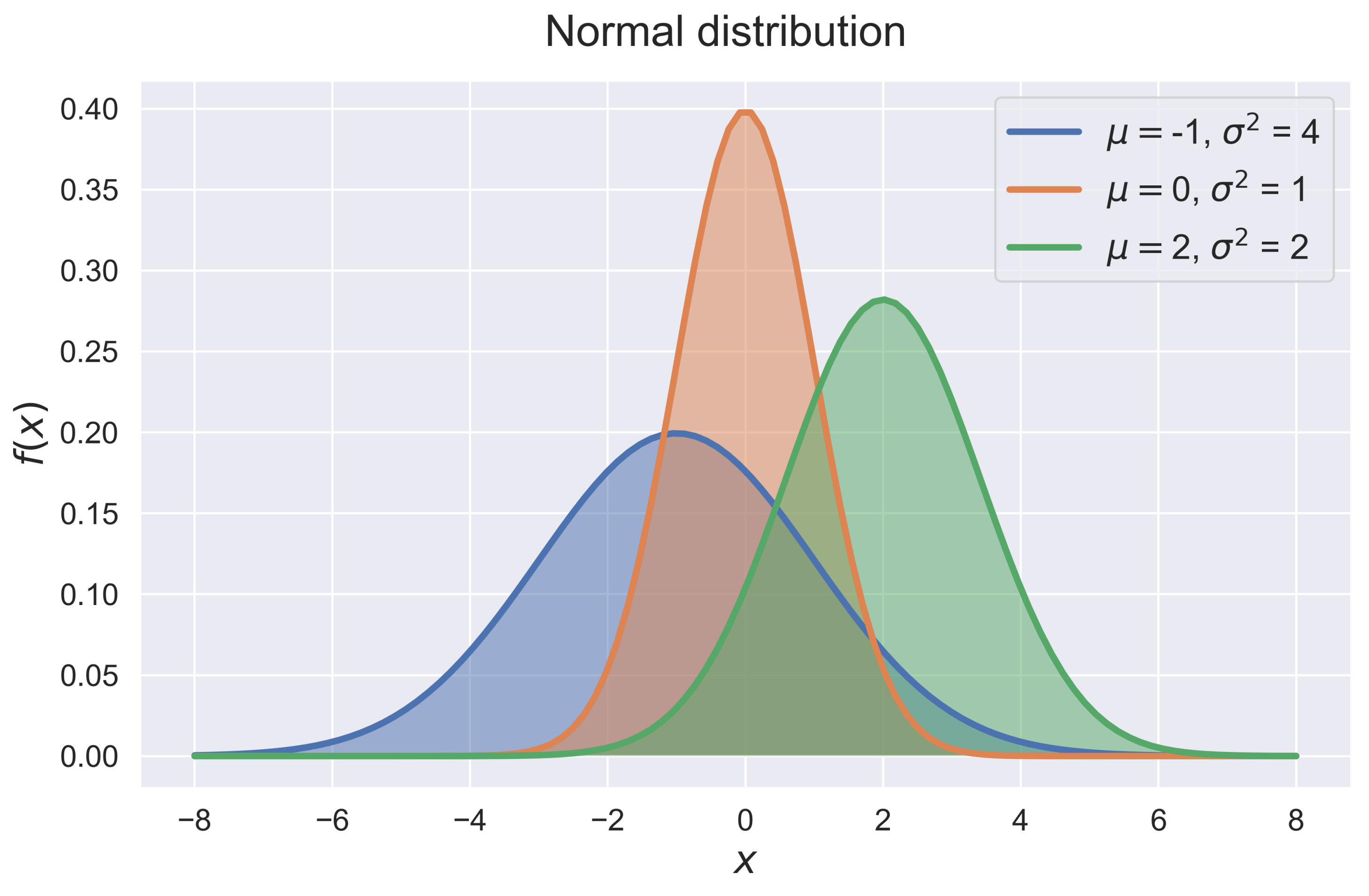
- Therefore we have

$$f(x_1, x_2, \cdots, x_n) = \frac{\partial^n F(x_1, x_2, \cdots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}.$$

Normal Distribution

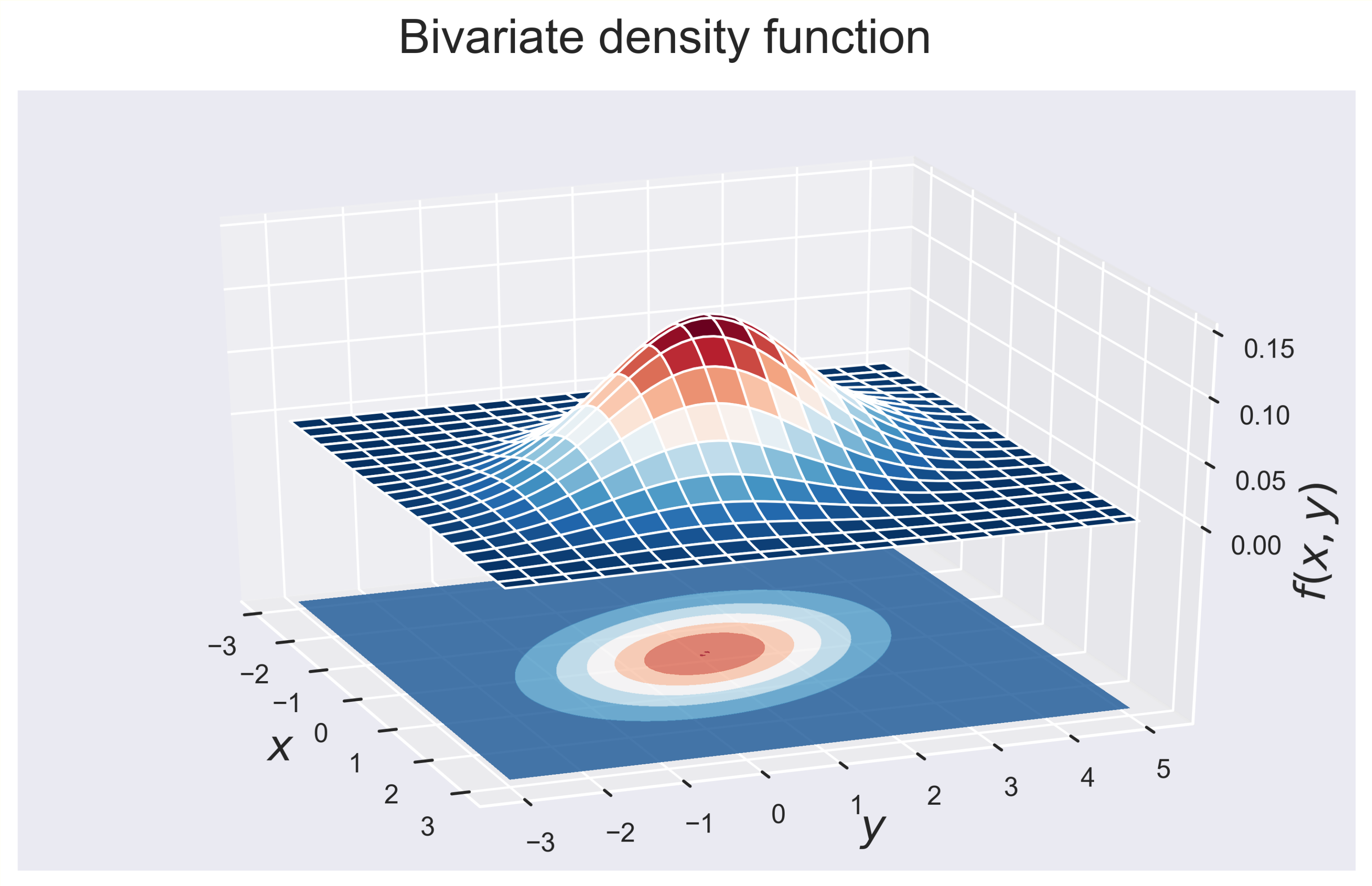
Single variate

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$



Bivariate

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$



INDEPENDENT RANDOM VARIABLES

For continuous random variables

Independent Random Variables

- Let $X_1, X_2, \cdots, X_n$ be continuous random variables with cumulative distribution functions $F_1(x), F_2(x), \cdots, F_n(x)$. And let $X = (X_1, X_2, \cdots, X_n)$.
- These random variables are **mutually independent** if the joint cumulative distribution of X is the product of individual cumulative distribution distributions $F_i(x_i)$,

$$F(x_1, x_2, \cdots, x_n) = F_1(x_1)F_2(x_2)\cdots F_n(x_n),$$

for any choice of $x_1, x_2, \cdots, x_n$.

Theorem

Let $X_1, X_2, \cdots, X_n$ be continuous random variables with density functions $f_1(x), f_2(x), \cdots, f_n(x)$.
Then these random variables are **mutually independent** if and only if $f(x_1, x_2, \cdots, x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$, for any choice of $x_1, x_2, \cdots, x_n$.

T

Independent Random Variables

Theorem

Let $X_1, X_2, \cdots, X_n$ be continuous random variables with density functions $f_1(x), f_2(x), \cdots, f_n(x)$.
Then these random variables are **mutually independent** if and only if $f(x_1, x_2, \cdots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$, for any choice of $x_1, x_2, \cdots, x_n$.

T

Theorem

Let $X_1, X_2, \cdots, X_n$ be mutually independent continuous random variables.
Let $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ be continuous functions.
Then $\phi_1(X_1), \phi_2(X_2), \cdots, \phi_n(X_n)$ are mutually independent.

T

Normal Distribution

Single variate

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Theorem

… these random variables are **mutually independent** if and only if $f(x_1, x_2, \cdots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$, for any choice of $x_1, x_2, \cdots, x_n$.

T

Bivariate

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$

How?

?

Normal Distribution

Single variate

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Theorem

… these random variables are **mutually independent** if and only if
$f(x_1, x_2, \cdots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$,
for any choice of $x_1, x_2, \cdots, x_n$.

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Bivariate

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$

How?

?

$\rho = \cdots$

=

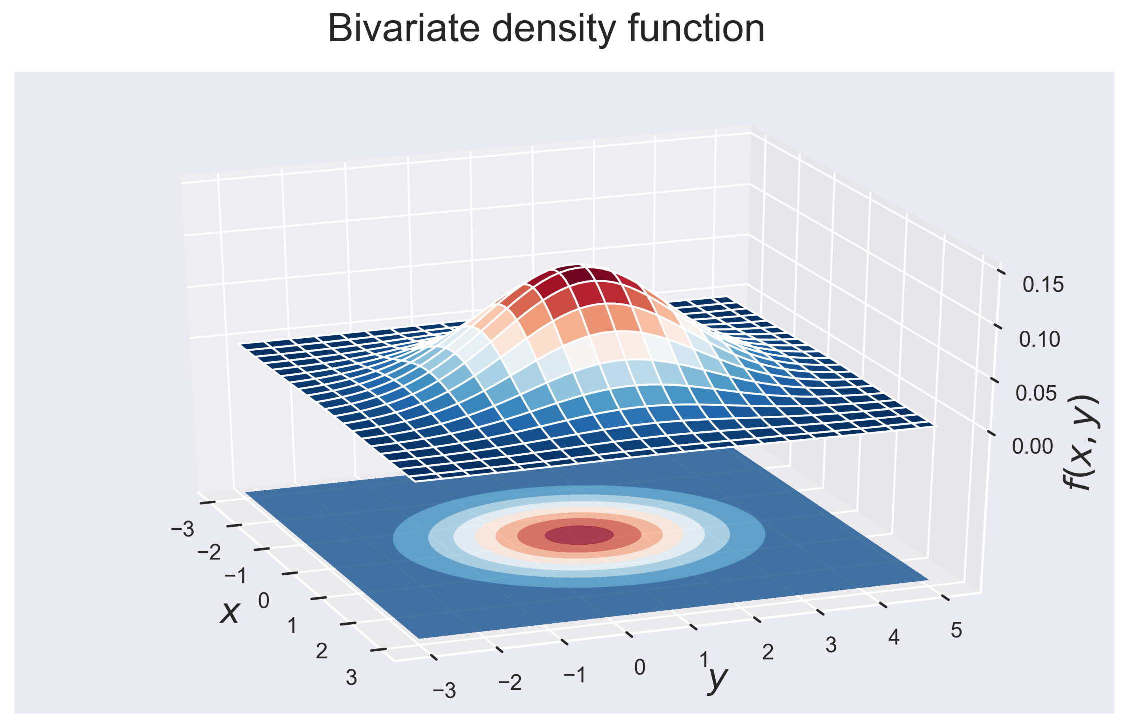
Normal Distribution

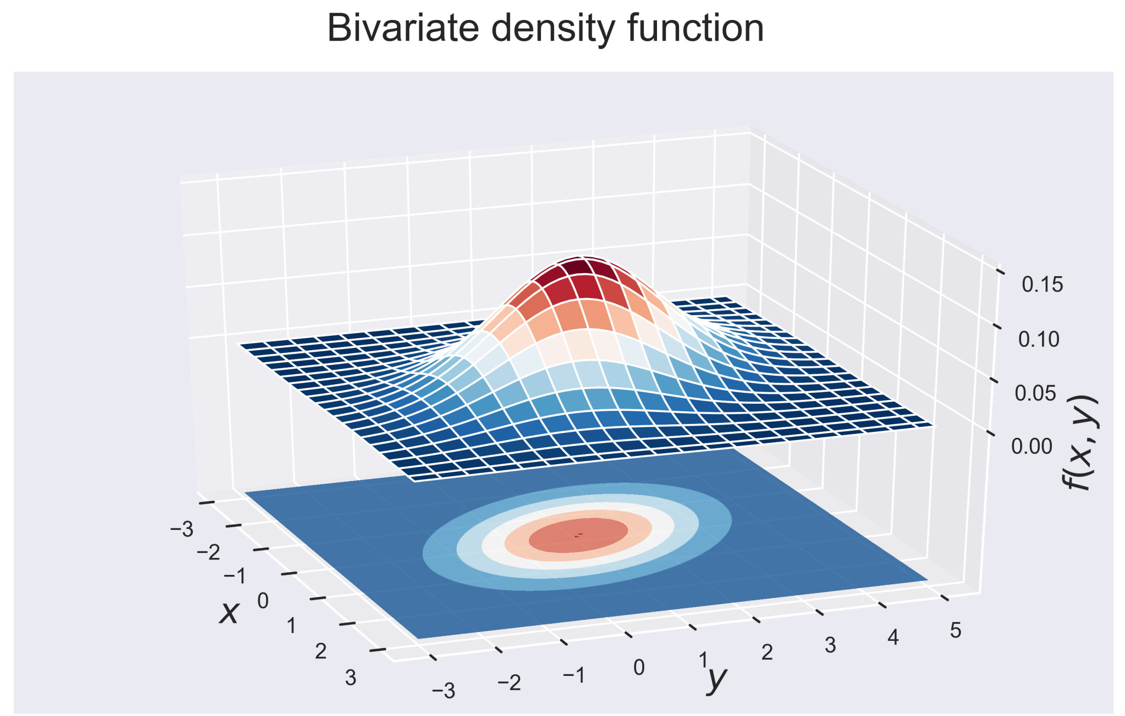
independent
$\rho = 0$

$=$

dependent
$\rho \neq 0$

$\neq$





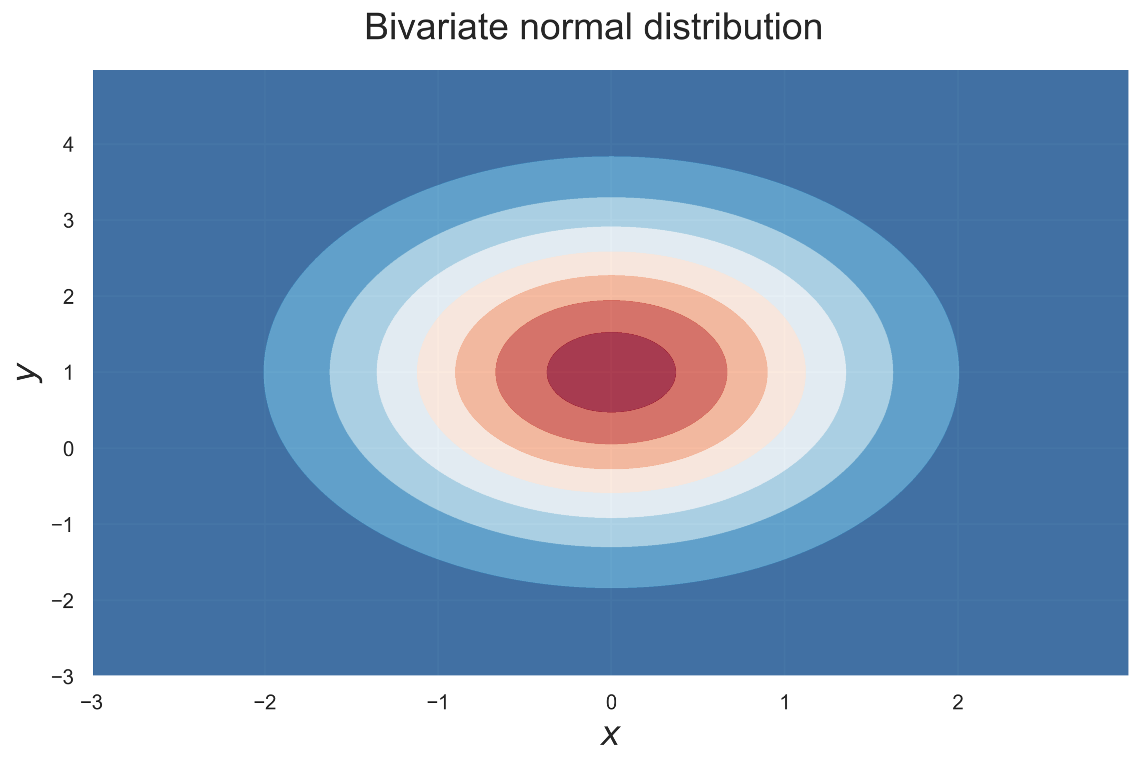
Normal Distribution

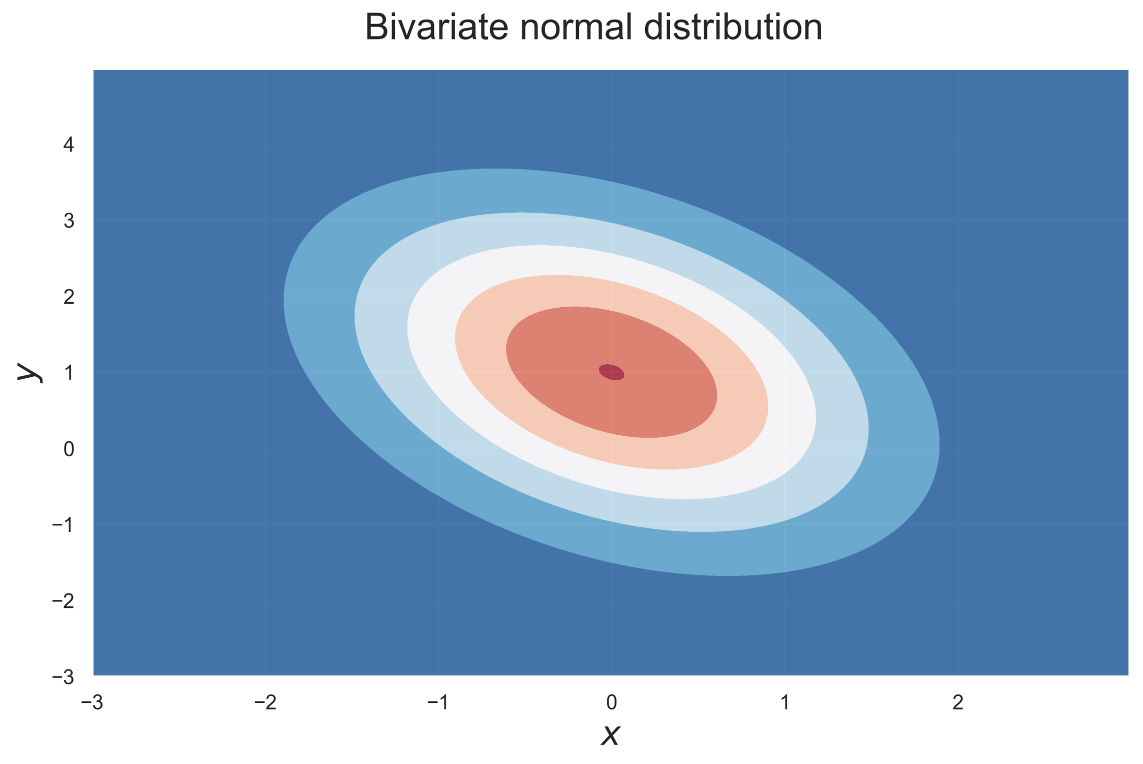
independent
$\rho = 0$

$=$

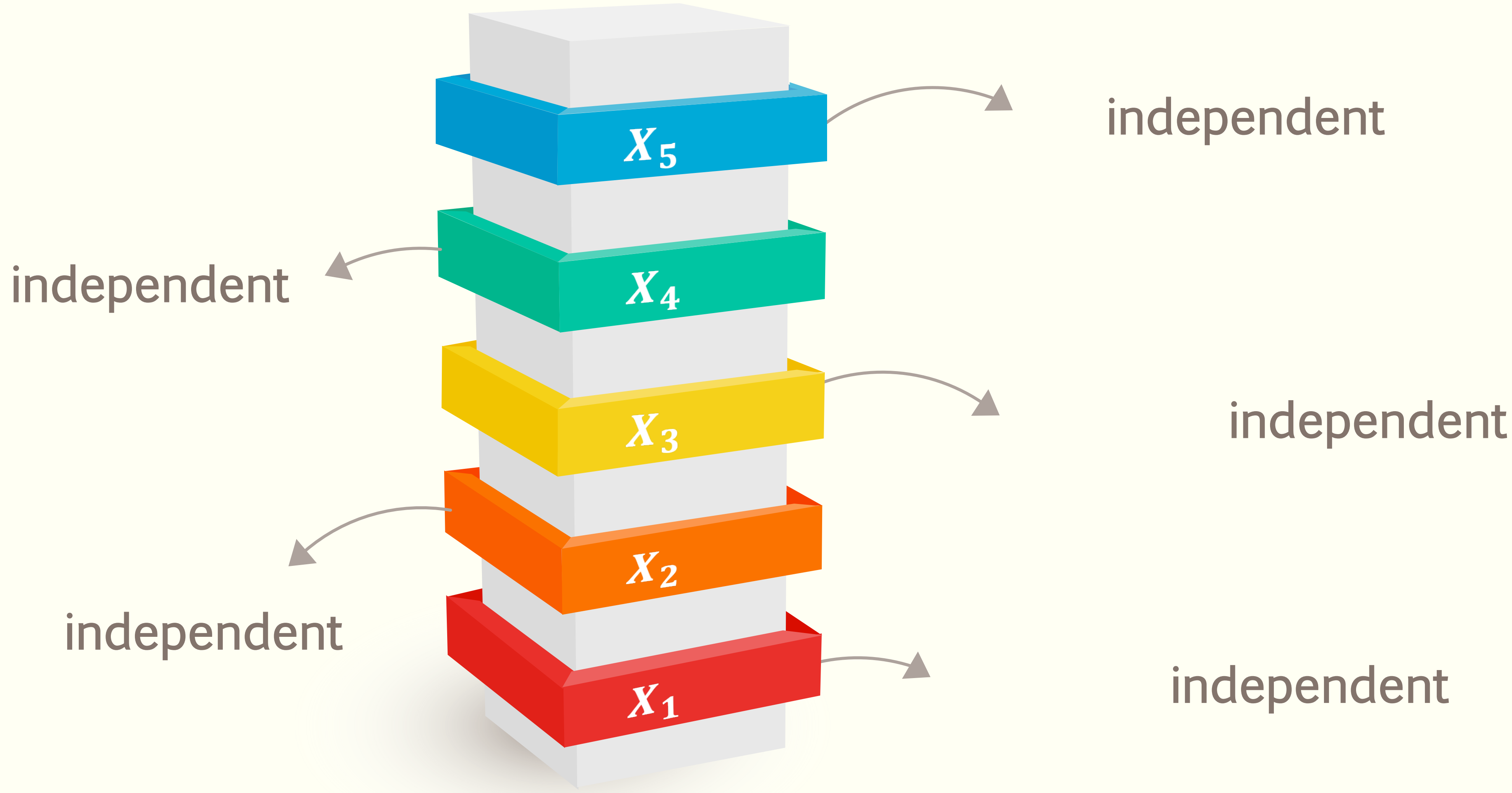
dependent
$\rho \neq 0$

$\neq$





Independent Random Variables



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Example

- Suppose you choose two numbers x and y, **independently** at random from the interval $[0, 1]$.
- **Given that** their sum lies in the interval $[0, 1]$, find the probability that

$$xy < \frac{1}{2}.$$

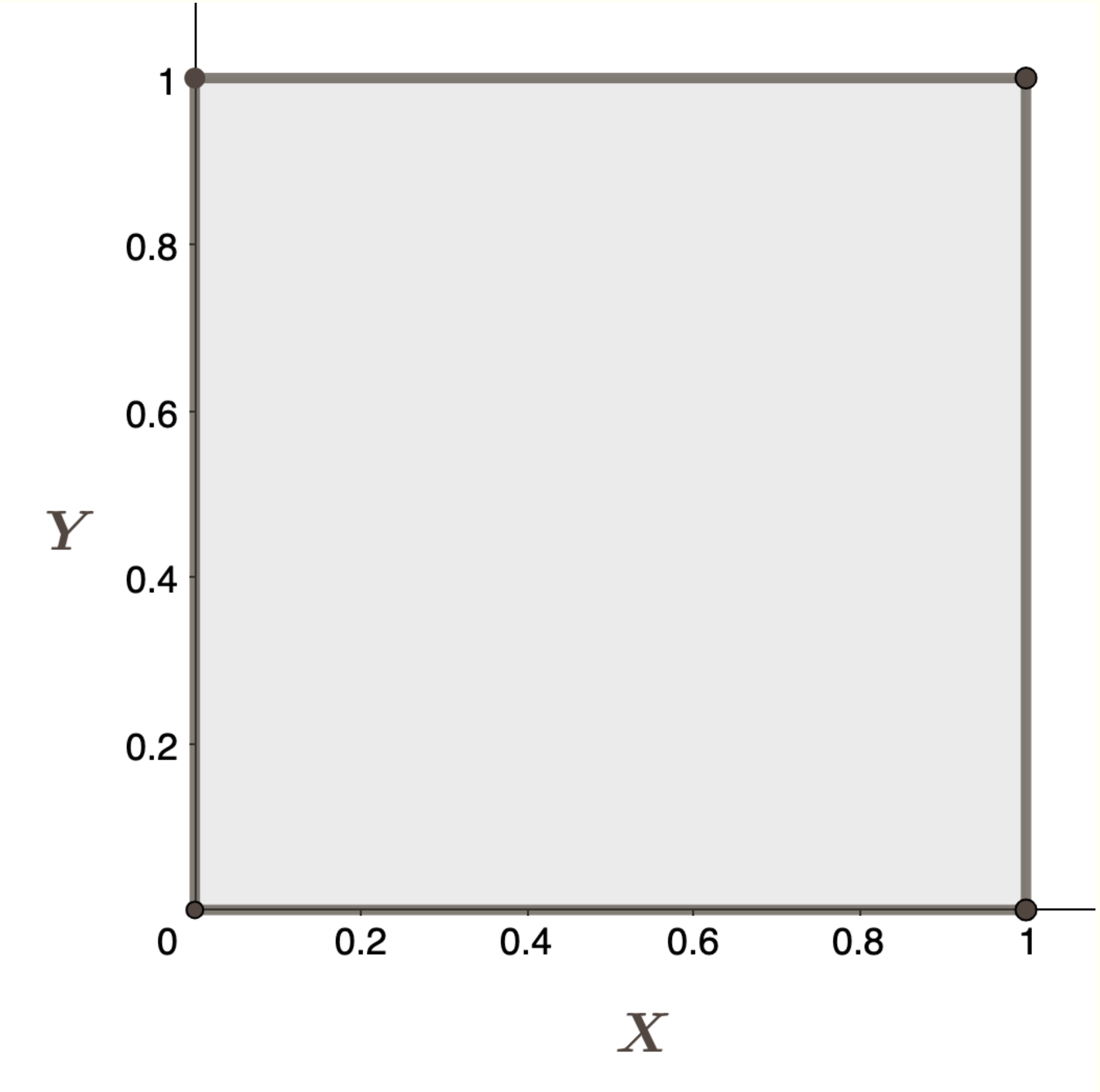
independently ?

given that ?

Example

independently

!



Example

independently

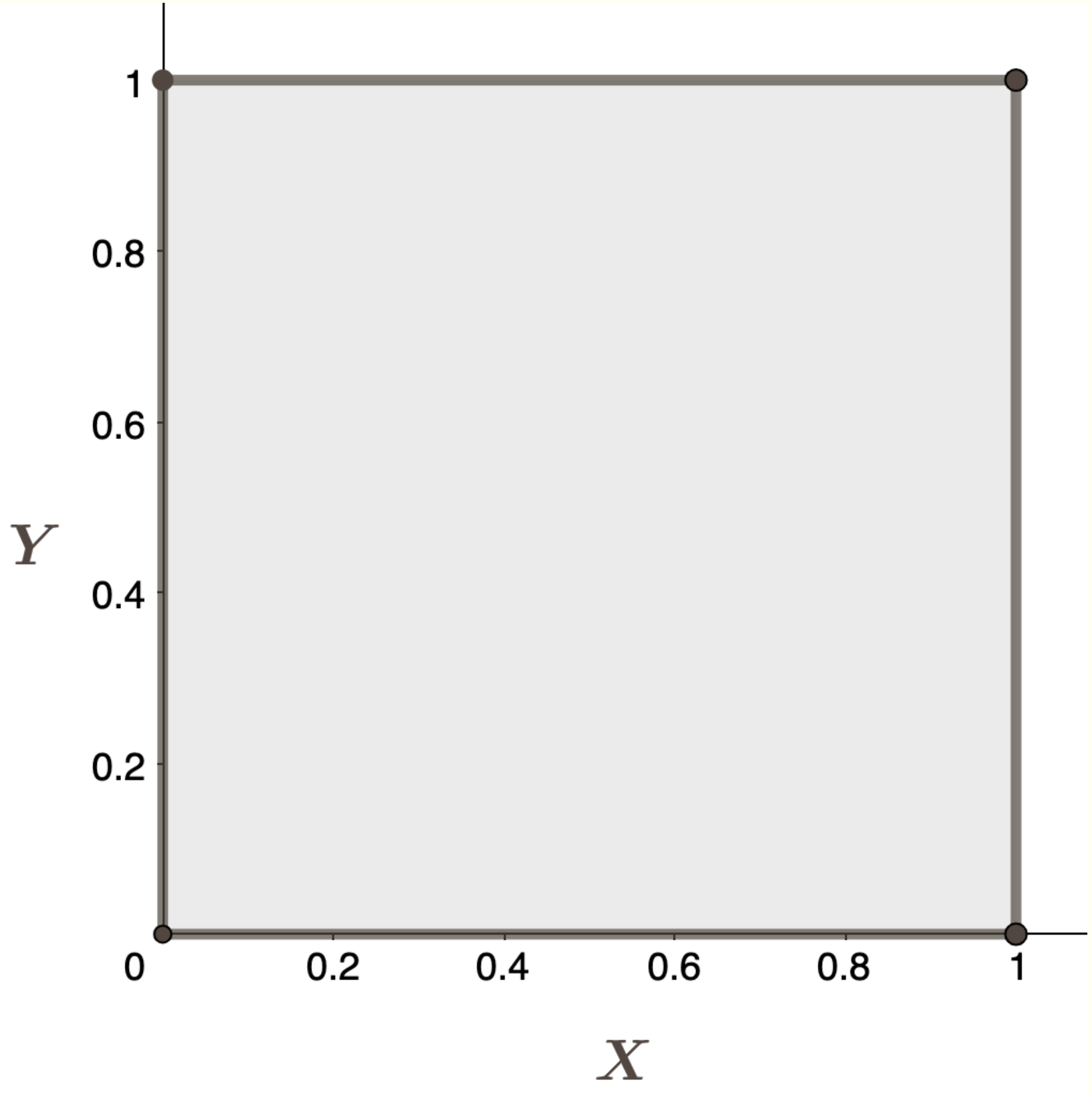
!

given that

!

Event E: $X + Y \in [0, 1]$

$$P(0 \le X + Y \le 1) = \cdots$$



Example

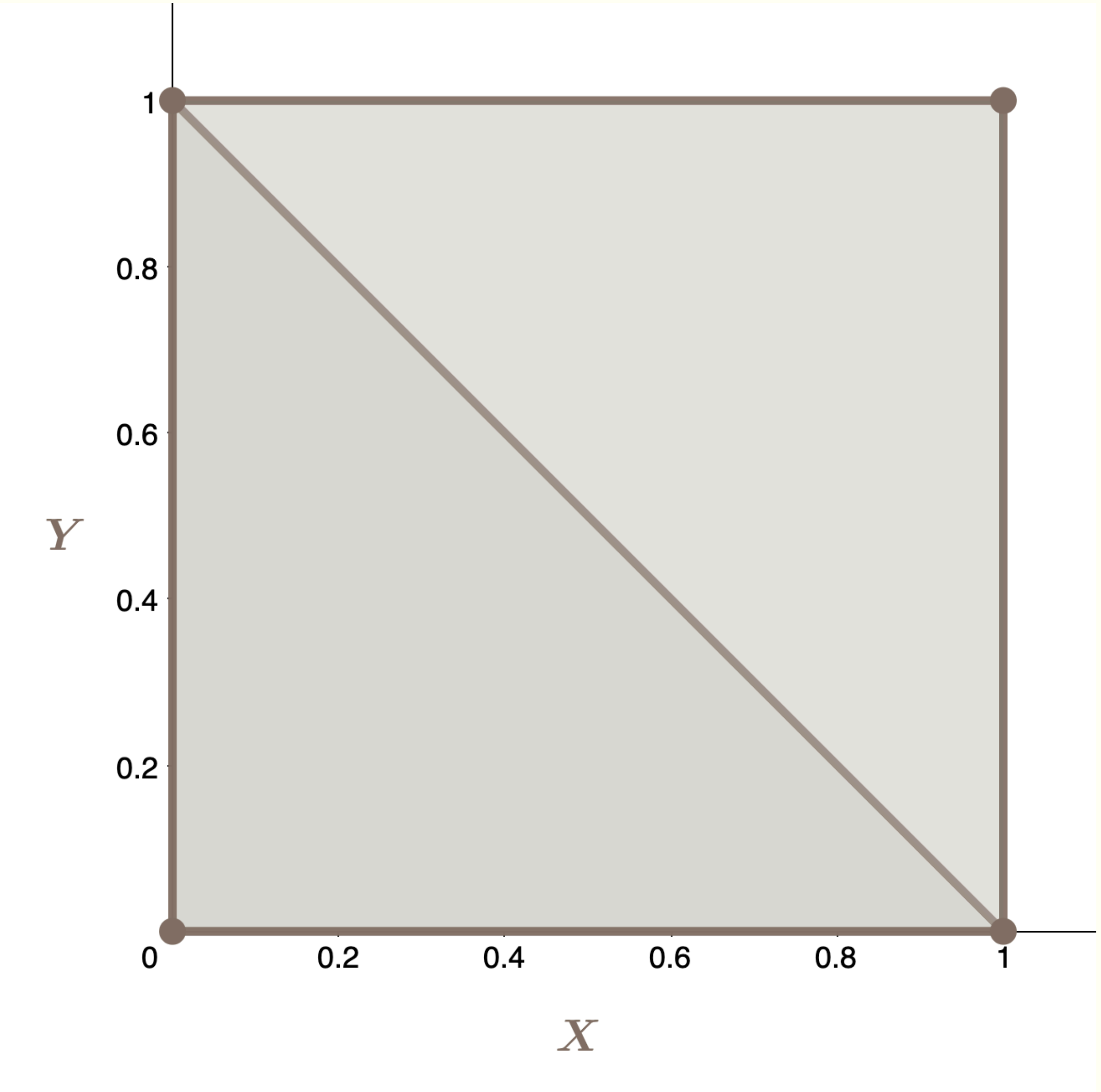
independently

!

given that

!

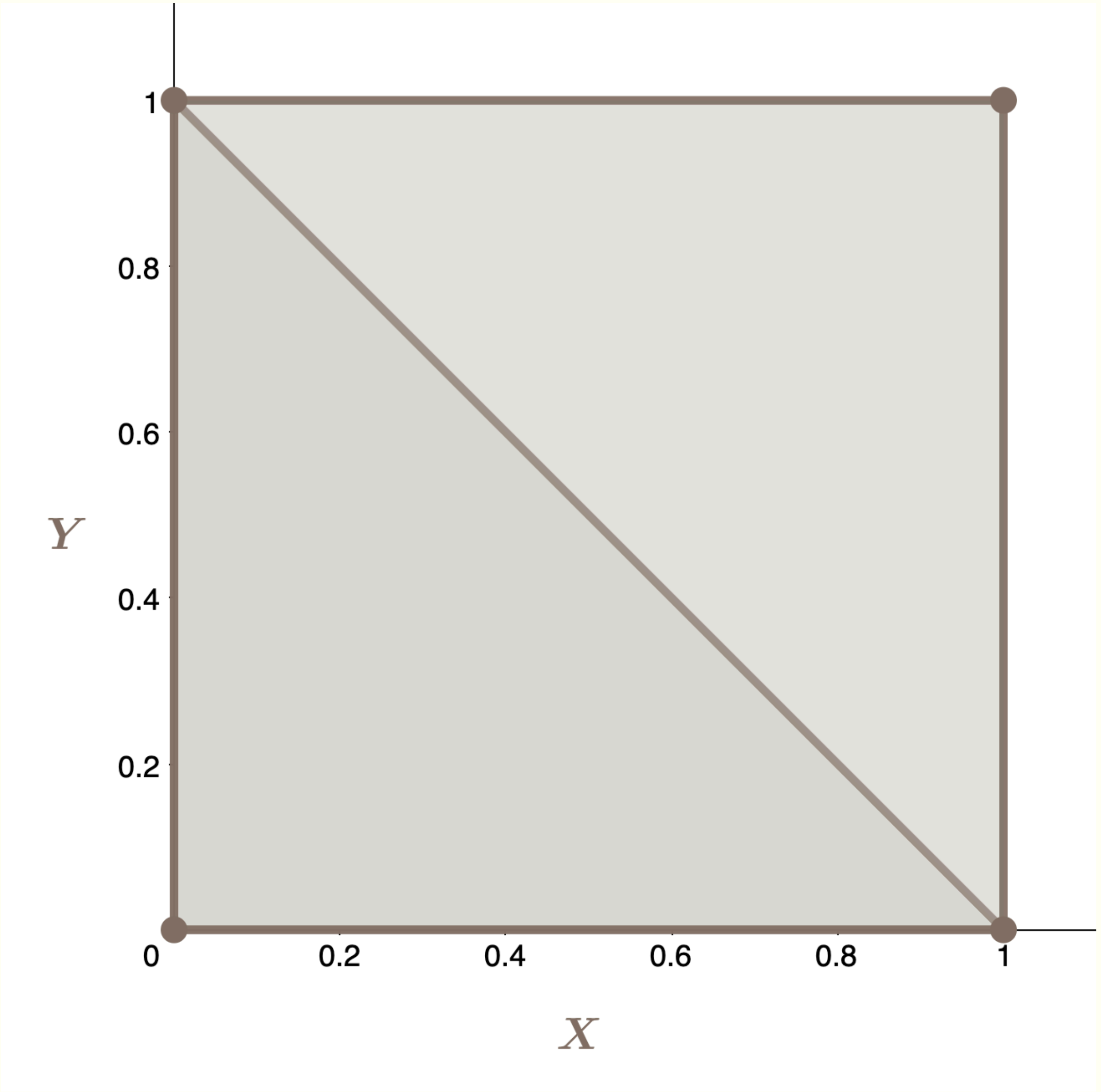
Event E: $\boldsymbol{X + Y \in [0, 1]}$
lines
$P(0 \leq X + Y \leq 1) = \frac{1}{2}$



Example

Event E: $\boldsymbol{X + Y \in [0, 1]}$
lines
$P(0 \le X + Y \le 1) = \frac{1}{2}$

Event $\boldsymbol{F}$: $\boldsymbol{XY < \frac{1}{2}}$
$P\left(XY < \frac{1}{2}\right)$



Example

Event E: $X + Y \in [0, 1]$

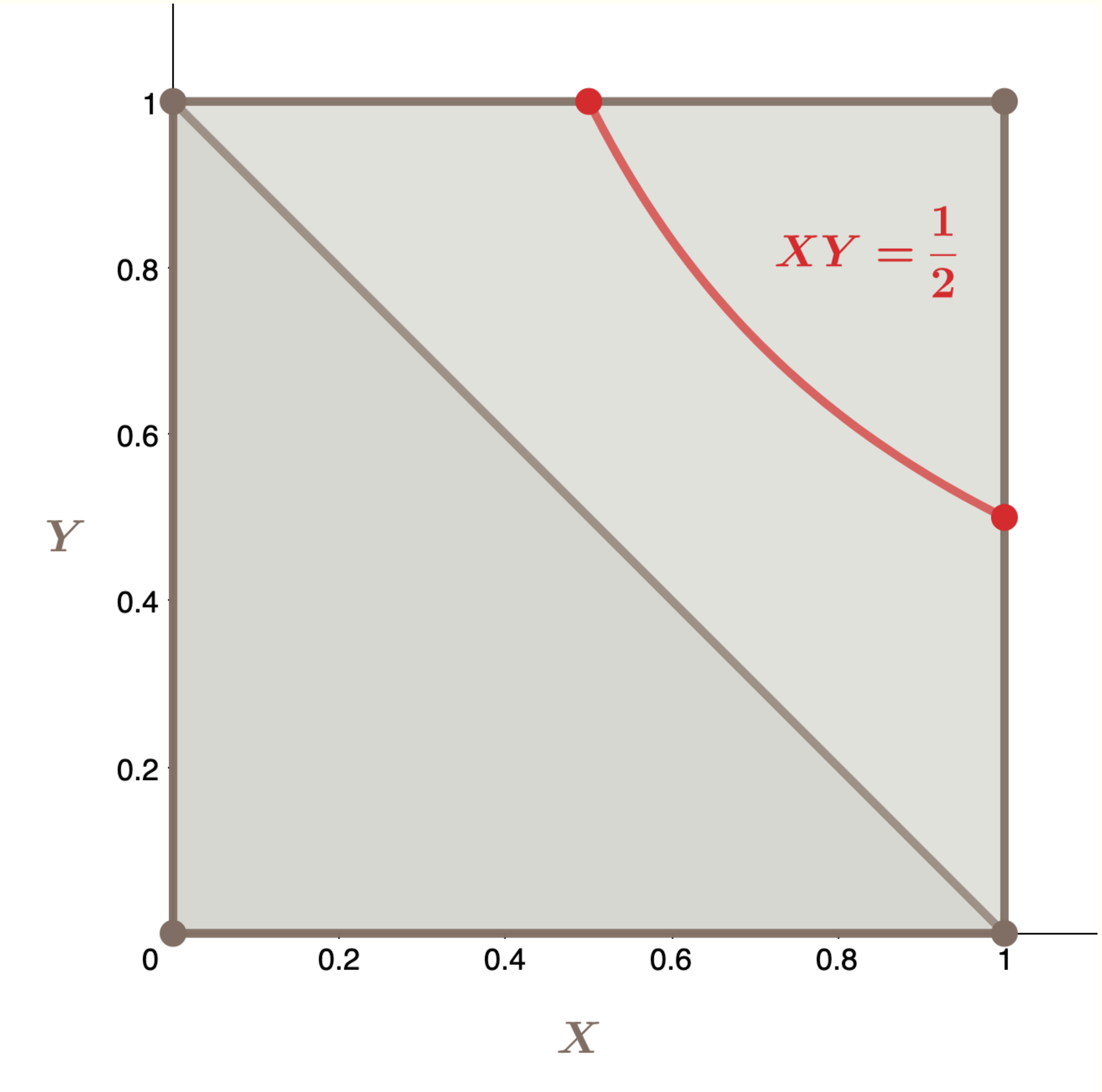
lines

$$P(0 \leq X + Y \leq 1) = \frac{1}{2}$$

Event F: $XY < \frac{1}{2}$

hyperbolas

$$P\left(XY < \frac{1}{2}\right)$$



Example

Event E: $X + Y \in [0, 1]$

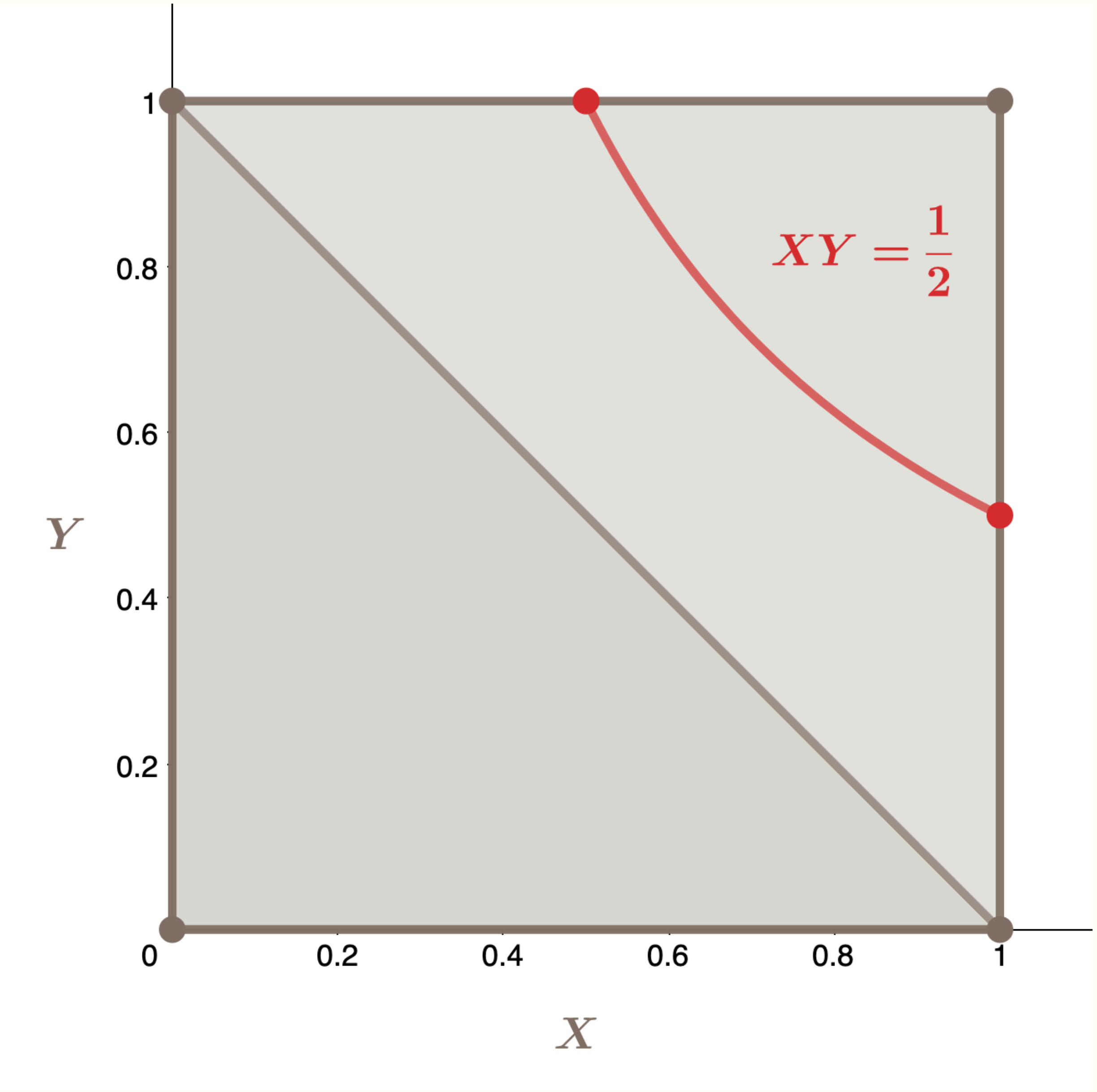
$$P(0 \le X + Y \le 1) = \frac{1}{2}$$

Event F: $XY < \frac{1}{2}$

$$P\left(XY < \frac{1}{2}\right)$$

Event $F|E$: $XY < \frac{1}{2} | X + Y \in [0, 1]$

$$P\left(XY < \frac{1}{2} \middle| 0 \le X + Y \le 1\right) = \cdots$$



Example

Event E: $X + Y \in [0, 1]$

$$P(0 \le X + Y \le 1) = \frac{1}{2}$$

Event F: $XY < \frac{1}{2}$

$$P\left(XY < \frac{1}{2}\right)$$

Event $F|E$: $XY < \frac{1}{2} | X + Y \in [0, 1]$

$$P\left(XY < \frac{1}{2} \middle| 0 \le X + Y \le 1\right) = 1$$