Math 22 Fall 2003
Final Exam

1. (20) Show that the vector $\begin{pmatrix} 3 \\ 9 \\ -4 \\ -6 \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \\ 3 \end{pmatrix}$. Find the weights, i.e., the numbers x_1, x_2, x_3 such that

$$\begin{pmatrix} 3 \\ 9 \\ -4 \\ -6 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -1 \\ 2 \\ 3 \end{pmatrix}.$$

2. (20) Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}.$$

(i) Find a basis for the row space Row A.

(ii) Find a basis for the column space Col A.

(iii) What is the dimension of Nul A?

3. (15) Let A and B be $n \times n$ matrices and let E be an elementary $n \times n$ matrix (i.e., a matrix obtained by applying one elementary row operation to I). Suppose $\det A = a$ and $\det B = b$. Express the following determinants in simplest form in terms of a and b:

(1) $\det(AB)$.

(2) $\det(2A)$.

(3) $\det(A^{-1})$, assuming A is invertible.

(4) $\det(A^{-1}BA^T)$, assuming A is invertible.

(5) $\det(EA)$, where E is the elementary matrix obtained by interchanging two rows of I.

(6) $\det(EA)$, where E is the elementary matrix obtained by adding a multiple of one row of I to another row of I.

(7) $\det(EA)$, where E is the elementary matrix obtained by multiplying a row of I by a scalar $r \neq 0$.

4. (25) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (-x_1 + 3x_2 + x_3, 2x_1 + 2x_2 + x_3, -x_1 - 5x_2 - 2x_3).$$

Let $\mathcal{B}$ be the basis of $\mathbf{R}^3$ consisting of $(1,0,0), (1,1,0)$ and $(1,1,1)$. Let $\mathcal{E}$ be the standard basis of $\mathbf{R}^3$ consisting of $(1,0,0), (0,1,0)$ and $(0,0,1)$.

(i) What is the matrix of T with respect to the basis $\mathcal{B}$ in the domain and $\mathcal{E}$ in the codomain?

(ii) If $\mathbf{v} = (3,-1,4)$, what is $[\mathbf{v}]_{\mathcal{B}}$?

Now consider the subspace V of $\mathbf{R}^3$ consisting of all (a,b,c) such that $a+b+c=0$. Note that the range (or image) of the linear transformation T is in V. Therefore there is a linear transformation $T' : \mathbf{R}^3 \to V$ defined by

$$T'(x_1, x_2, x_3) = (-x_1 + 3x_2 + x_3, 2x_1 + 2x_2 + x_3, -x_1 - 5x_2 - 2x_3).$$

Let $(-1,3,-2)$ and $(-2,1,1,)$ be a basis $\mathcal{C}$ for V.

(iii) Find the matrix of T' with respect to the basis $\mathcal{B}$ of $\mathbf{R}^3$ and $\mathcal{C}$ of V.

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5. (20) Given a matrix

$$A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix}.$$

(i) Find the eigenvalues of A.

(ii) Find an eigenvector for each eigenvalue of A.

(iii) Find a 2×2 invertible matrix P and a 2×2 diagonal matrix D such that $A = PDP^{-1}$.

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6. (20) Consider the line L in $\mathbf{R}^3$ through the origin which is given by $t(2,-1,2)$ for all scalars t. Find the distance between L and the point $(1,1,1)$.

7. (30) Short answer problems.

(1) Let A be an $m \times n$ matrix and consider the linear system of equations $A\mathbf{x} = \mathbf{b}$.

(a) Give conditions on m and n such that there is not a solution for every $\mathbf{b} \in \mathbf{R^m}$.

(b) Assume that there is a solution for every $\mathbf{b} \in \mathbf{R^m}$. Give conditions on m and n such each solution is not unique.

(2) What are all possible 2×2 matrices in reduced echelon form?

(3) Let A be an $m \times n$ matrix and assume that $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a non-trivial solution to $A\mathbf{x} = \mathbf{0}$. What can you say about the columns of A considered as vectors in $\mathbf{R^m}$? (This should be a statement about the vectors, not about the matrix.)

(4) Let a be a fixed real number and consider V_a, the set of all polynomials of degree $\leq n$ whose constant term is a, i.e., all $a + a_1x + \cdots + a_nx^n$, where $a_1, \ldots, a_n$ are any scalars. For which a is V_a a subspace of P_n, the vector space of all polynomials of degree $\leq n$? What is its dimension?

(5) Give two statements which are equivalent to the statement 'The $n \times n$ matrix A is invertible.'

(6) When is the diagonal matrix

$$\begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}$$

invertible and what is its inverse?

(7) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors in $\mathbf{R}^3$ and the angle between them is $\frac{\pi}{4}(= 45°)$, then $\mathbf{u} \cdot \mathbf{v} =$

(8) Complete the following sentence: If P is an $r \times r$ transition matrix which represents a Markov chain and $\mathbf{v}$ is an $r-$dimensional probability vector whose jth entry is 1, then the ith entry of the probability vector $P^{101}\mathbf{v}$ is the probability that