Math 22 Fall 2003
Second Hour Exam

1. (20) Consider the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^{2}$ 3 defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_3 + x_4, -3x_1 + 2x_2 - x_4, 3x_2 + 9x_3 + 3x_4).$$

Let A be the matrix of T (i.e., $T(\mathrm{x}) = A\mathrm{x}$).

(i) Find A. (A mistake here will affect the rest of the problem.)

(ii) Find a basis for Col A.

(iii) Find a basis for Row A.

(iv) What is the dimension of the kernel of T? (No details necessary.)

(v) Is T onto? Give a reason for your answer.

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2. (20) Let V be a two dimensional vector space with basis $\mathcal{B} = \{v_1, v_2\}$. Let a be a fixed scalar and let $T : V \to V$ be a linear transformation such that $T(v_1) = av_2$ and $T(v_2) = av_1$.

(i) What is the matrix $[T]_{\mathcal{B}}$?

(ii) Is $[T]_{\mathcal{B}}$ diagonalizable? Give reasons for your answer.

(iii) If $v \in V$ and $[v]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, what is $[T(v)]_{\mathcal{B}}$?

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3. (20) Find a basis for all vectors of the form

the subspace of $\mathbb{R}^4$

$$\begin{pmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{pmatrix},$$

for [redacted] $a, b, c \in \mathbf{R}$.

4. (20) Consider the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Find all eigenvalues and a basis for each eigenspace.

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5. (30) True - False. In each of the following, circle T if the statement is always true; circle F otherwise.

(a). If $\{v_1, \ldots, v_k\}$ is a linearly independent set of vectors in a vector space V, then every vector in Span$\{v_1, \ldots, v_k\}$ can be written in exactly one way as a linear combination of $v_1, \ldots, v_k$.

(b). If the $n \times n$ matrices A and B are both similar to an $n \times n$ matrix C, then A is similar to B.

(c). Col A is the set of all vectors that can be written as $A\mathbf{x}$ for some $\mathbf{x}$.

(d). If the nullspace of a 5×6 matrix A is 4-dimensional, then Col A is 1 dimensional.

(e). Col $A = $ Row A^T, for any matrix A.

(f). If A is a 7×5 matrix, then the largest possible rank of A is 5.

(g). If 0 is an eigenvalue of an $n \times n$ matrix A, then A is not invertible.

(h). If a 4×4 matrix A has exactly 3 distinct eigenvalues, then A is not diagonalizable.

(i). The set of all eigenvectors of an $n \times n$ matrix A is a subspace of $\mathbf{R^n}$.

(j). If A, P and D are $n \times n$ matices such that P is invertible, D is diagonal and $A = PDP^{-1}$ then the columns of P are eigenvectors of A.