MATH 22 LINEAR ALGEBRA

ANSWER KEY FOR HOMEWORKS DUE 9/29/04 AND 10/6/04

1.1: 4, 12, 16, 18, 20, 32

(4.) $$\begin{bmatrix} 1 & -5 & 1 \\ 3 & -7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & \frac{1}{4} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 9/4 \\ 0 & 1 & 1/4 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 9/4 \\ x_2 = 1/4 \end{cases}$$

CHECK: $1\left(\frac{9}{4}\right) - 5\left(\frac{1}{4}\right) = 1$
$3\left(\frac{9}{4}\right) - 7\left(\frac{1}{4}\right) = 5$

ALSO, $\left(\frac{9}{4}, \frac{1}{4}\right)$ CAN BE FOUND AS THE INTERSECTION OF THE GRAPHS OF THE TWO LINES.

(12.) $$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

THE SYSTEM IS INCONSISTENT, THERE ARE NO SOLUTIONS.

(16.) $$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

YES, THE SYSTEM IS CONSISTENT (WITH INFINITELY MANY SOLUTIONS.)

(18.) $$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

THE SYSTEM IS INCONSISTENT, SO <u>NO</u>, THE THREE LINES HAVE NO COMMON POINT OF INTERSECTION. THIS CAN BE VERIFIED BY DRAWING THE GRAPHS OF THE THREE LINES.

NEXT ↗

(CONT'D)

(20.) $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -3 \\ 0 & 2h+4 & 0 \end{bmatrix}$ THE SYSTEM IS CONSISTENT FOR ALL h.

(32.) REPLACEMENT : $R_3 \rightarrow R_3 \pm 3R_2$

1.2 : 2, 4, 6, 10, 16, 20, 24

(2.) (a.) REDUCED ECHELON FORM

(b.) ECHELON FORM

(c.) NEITHER

(d.) ECHELON FORM

(4.) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ PIVOT COLUMNS : 1, 2, 4

(6.) $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(10.) $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

<u>BASIC VARIABLES</u> : x_1, x_3

<u>FREE VARIABLES</u> : x_2

$$\begin{cases} x_1 = 2x_2 - 4 \\ x_3 = -7 \end{cases}$$

<u>NEXT</u> ↗

(CONT'D)

(16.) (a.) CONSISTENT, UNIQUE (ONE SOLUTION)
(b.) CONSISTENT, NOT UNIQUE (INFINITE SOLUTIONS)

(20.) THERE ARE INFINITELY MANY CORRECT ANSWERS TO THIS PROBLEM.

(24.) No. THE SYSTEM IS NOT CONSISTENT BY THEOREM 2.

1.2: 12, 14, 28, 31

(12.) $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & -8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

BASIC VARIABLES: x_1, x_3

FREE VARIABLES: x_2, x_4

$$\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_3 = -3 + 2x_4 \\ x_2, x_4 \text{ FREE} \end{cases}$$

(CONSISTENT, INFINITE SOLUTIONS.)

(14.) $$\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

BASIC VARIABLES: x_1, x_2, x_5

FREE VARIABLES: x_3, x_4

$$\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_5 = 0 \\ x_3, x_4 \text{ FREE} \end{cases}$$

(CONSISTENT, INFINITE SOLUTIONS.)

NEXT ↗

(CONT'D)

(28.) ALL COLUMNS BUT THE LAST COLUMN MUST BE PIVOT COLUMNS. (OR ELSE THE SYSTEM IS INCONSISTENT OR HAS A FREE VARIABLE, IN WHICH CASE THERE IS NOT A UNIQUE SOLUTION.)

(31.) YES. AN OVERDETERMINED SYSTEM CAN BE CONSISTENT. CONSIDER THE FOLLOWING EXAMPLES WITH THREE EQUATIONS AND TWO UNKNOWNS:

EXAMPLE I:

$$\begin{cases} x+y = 4 \\ 2x+2y = 8 \\ x-y = 0 \end{cases} \qquad \begin{cases} x = 2 \\ y = 2 \end{cases}$$

EXAMPLE I:

$$\begin{cases} y = x \\ y = 2x \\ y = 3x \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \end{cases}$$

1.3: 8, 10, 12, 14, 18, 26, 30

(8.) $w = (-1)u + 2v$
$x = (-2)u + 2v$
$y = (-2)u + 3.5v$
$z = (-3)u + 4v$

(10.) $x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

NEXT ↗

(CONT'D)

(12.) $\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

<u>NO</u>, b IS NOT A LINEAR COMBINATION OF a_1, a_2, a_3 BECAUSE THE SYSTEM IS INCONSISTENT.

(14.) $\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix}$

<u>YES</u>, b IS A LINEAR COMBINATION OF THE VECTORS FORMED FROM THE COLUMNS OF THE MATRIX A.

(18.) $h = -\frac{7}{2}$

(26.) (a.) $\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

<u>YES</u>. $b \in W$.

(b.) $\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \in W$ (OBVIOUS.)

(30.) <u>YES</u>. $v = \frac{1}{m}(m_1 v_1 + \dots + m_k v_k)$

$= \frac{m_1}{m} v_1 + \dots + \frac{m_k}{m} v_k$

THUS v IS A LINEAR COMBINATION OF $v_1, \dots, v_k$ SO $v \in \text{SPAN}\{v_1, \dots, v_k\}$.

<u>NEXT</u> ↗

1.4 : 8, 12, 14, 16, 18, 22, 32

(8.) $\begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

(12.) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\Rightarrow \begin{cases} x_1 = 3/5 \\ x_2 = -4/5 \\ x_3 = 1 \end{cases} \Rightarrow x = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}.$

CHECK : $\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$

(14.) $\begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 15 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 7 & -7 & 8 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 29 \end{bmatrix}$ INCONSISTENT $\Rightarrow$ NO u IS NOT IN THE SUBSET OF $\mathbb{R}^3$ SPANNED BY THE COLUMNS OF A.

(16.) $\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 0 & 14 & 12 & -5b_1+b_3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$ CONSISTENT IFF $b_1 + 2b_2 + b_3 = 0$

So, $Ax = b$ DOES NOT HAVE A SOLUTION WHEN $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ FOR EXAMPLE.

$Ax = b$ DOES HAVE A SOLUTION IFF $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ LIES IN THE PLANE $b_1 + 2b_2 + b_3 = 0$.

(CONT'D)

(18.) $\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 1 & 1 & -5 \\ -2 & -8 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ -2 & -8 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B DOES NOT HAVE A PIVOT POSITION IN THE 4TH ROW, SO BY THEOREM 4, THE ANSWER TO BOTH QUESTIONS IS NO.

(22.) $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$

THUS THE 3×3 MATRIX $[v_1 \; v_2 \; v_3]$ HAS A PIVOT POSITION IN EVERY ROW, SO BY THEOREM 4 THE ANSWER IS YES, $\text{SPAN}\{v_1, v_2, v_3\} = \mathbb{R}^3$.

(32.) NO. BY WAY OF CONTRADICTION, SUPPOSE $n < m$ AND n VECTORS IN $\mathbb{R}^m$ SPAN $\mathbb{R}^m$. BY THEOREM 4, THE $m\times n$ MATRIX FORMED BY THE n VECTORS HAS A PIVOT POSITION IN EVERY ROW, AND THUS HAS m PIVOT POSITIONS. THIS IS A CONTRADICTION BECAUSE THERE CAN BE AT MOST n PIVOT POSITIONS IN A $m\times n$ MATRIX.

1.5: 6, 14, 28

(6.) $\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -4x_3 \\ x_2 = 3x_3 \\ x_3 \text{ FREE} \end{cases}$

$\Rightarrow x = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ SO THE SOLUTION SET IS A LINE IN $\mathbb{R}^3$.

(14.) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ 8+x_4 \\ 2-5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}$ A "LINE" IN $\mathbb{R}^4$

PASSING THROUGH $\begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix}$ IN THE DIRECTION $\begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}$.

(28.) <u>No</u>. IF $b \neq \vec{0}$ THEN THE SOLUTION SET OF $Ax = b$ CANNOT BE A PLANE THROUGH THE ORIGIN, BECAUSE IF THIS WERE THE CASE, WE WOULD HAVE $A\vec{0} = b$ BUT $A\vec{0} = \vec{0}$ AND THUS $b = \vec{0}$ WHICH CONTRADICTS THE FACT THAT $b \neq 0$.