MATH 22 LINEAR ALGEBRA FALL '04

ANSWER KEY FOR HOMEWORK DUE 10/13/04

1.5: 16, 18, 30, 36

(16.) $\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

BASIC VARIABLES: x_1, x_2

FREE VARIABLES: x_3

$$\begin{cases} x_1 = -4x_3 - 5 \\ x_2 = 3x_3 + 3 \\ x_3 = x_3 \end{cases} \Rightarrow x = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

THIS IS AN EXAMPLE OF THEOREM 6 WHERE $p = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ IS A PARTICULAR SOLUTION OF THE NONHOMOGENEOUS SYSTEM AND $V_h = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ IS THE SOLUTION SET OF THE HOMOGENEOUS SYSTEM. THE SOLUTION SET IS A LINE IN $\mathbb{R}^3$ PASSING THROUGH $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ IN THE DIRECTION $\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$. THIS LINE DOES NOT PASS THROUGH THE ORIGIN.

(18.) $x_1 - 3x_2 + 5x_3 = 0 \Rightarrow x_1 = 3x_2 - 5x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

SO THE SOLUTION SET IS A PLANE PASSING THROUGH THE ORIGIN. BY THEOREM 6, THE SOLUTION SET OF $x_1 - 3x_2 + 5x_3 = 4$ IS $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ SINCE $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ IS A PARTICULAR SOLUTION. (WE COULD ALSO USE $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ INSTEAD OF $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ SINCE $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ IS ANOTHER PARTICULAR SOLUTION.) THE SOLUTION SET IS A PLANE PARALLEL TO THE PREVIOUS (HOMOGENEOUS) SOLUTION SET, BUT SHIFTED AND NOT PASSING THROUGH THE ORIGIN.

(30.) (a.) YES, BECAUSE THERE IS ONE FREE VARIABLE.
(b.) NO, BY THEOREM 4.

(36.) THERE ARE INFINITELY MANY SUCH MATRICES.
FOR EXAMPLE, LET $A = \begin{bmatrix} k & k & k \\ k & k & k \\ k & k & k \end{bmatrix}$, $k \neq 0$.

FOR ANOTHER EXAMPLE, LET $A = \begin{bmatrix} k & k & k \\ ak & ak & ak \\ bk & bk & bk \end{bmatrix}$, $k \neq 0$.

1.7: 2, 10, 16, 24, 28

(2.) $\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

THE ONLY SOLUTION TO THE HOMOGENEOUS SYSTEM IS THE TRIVIAL SOLUTION, SO THE VECTORS ARE LINEARLY INDEPENDENT BY THEOREM P. 66.

(10.) (a.) $\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}$ INCONSISTENT, SO $v_3 \notin \text{Span}\{v_1, v_2\}$ FOR ALL $h \in \mathbb{R}$.

(b.) $\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

SO $\{v_1, v_2, v_3\}$ IS LINEARLY DEPENDENT FOR ALL $h \in \mathbb{R}$, BY THEOREM P. 66.

(16.) NO, THE VECTORS ARE LINEARLY DEPENDENT BY THEOREM P. 67 BECAUSE EACH VECTOR IS A MULTIPLE OF THE OTHER.

(24.) BY THEOREM P. 66 THERE IS AT MOST ONE PIVOT POSITION, SO THE THREE POSSIBLE ECHELON FORMS ARE

$$\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

EXAMPLES:

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(28.) 5 (BY THEOREM 4, SINCE THE NUMBER OF PIVOT COLUMNS EQUALS THE NUMBER OF PIVOT ROWS EQUALS THE NUMBER OF PIVOT POSITIONS.)

1.7: 8, 14, 18, 32

(8.)
$$\begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ -3 & 7 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

THE COLUMNS ARE LINEARLY <u>DEPENDENT</u> BY THEOREM P. 66.
MORE GENERALLY, n VECTORS IN $\mathbb{R}^m$ ARE LINEARLY DEPENDENT WHENEVER $n > m$. (BY THEOREM 8.)

(14.)
$$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

SO THE VECTORS ARE LINEARLY <u>DEPENDENT</u> IFF $h = -10$ BY THEOREM P. 66.

(18.) $\begin{bmatrix} 4 & -1 & 2 & 8 & 0 \\ 4 & 3 & 5 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -1 & 2 & 8 & 0 \\ 0 & -4 & -3 & 7 & 0 \end{bmatrix}$

SO THE VECTORS ARE LINEARLY <u>DEPENDENT</u> BY THEOREM P. 66. (AGAIN, n VECTORS IN $\mathbb{R}^m$ ARE LINEARLY DEPENDENT IF $n > m$ BY THEOREM 8.)

(32.) $x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ <u>CHECK</u>: $\begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

1.8: 6, 10, 12, 20, 32

(6.) $\begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

<u>BASIC VARIABLES</u>: x_1, x_2

<u>FREE VARIABLES</u>: x_3

$$\begin{cases} x_1 = -3x_3 + 7 \\ x_2 = -x_3 + 3 \\ x_3 = x_3 \end{cases} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 + 7 \\ -x_3 + 3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

THUS x IS <u>NOT</u> UNIQUE.

(10.) $\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 3 & 6 & 6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ <u>BASIC VARIABLES</u>: x_1, x_2, x_4

<u>FREE VARIABLES</u>: x_3

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

<u>CHECK</u>: $\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}\left(x_3\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}\right) = x_3\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}.$

(12.) $\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 3 & 6 & 6 & -4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 3 & 6 & 6 & -4 \\ 0 & 3 & 6 & 9 & -3 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 3 & 6 & 6 & -4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 3 & 6 & 6 & -4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix}$ THE SYSTEM IS INCONSISTENT, SO THE ANSWER IS <u>NO</u>.

(20.) $T(x) = x_1 v_1 + x_2 v_2 = x_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} x$ SO $A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$.

(32.) $T : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, x_2) \mapsto (4x_1 - 2x_2, 3|x_2|)$

$T(1,1) = (2,3)$

$T(-1,-1) = (-2,3)$

$T((1,1) + (-1,-1)) = T(0,0) = (0,0) \neq (0,6) = T(1,1) + T(-1,-1)$

THUS T IS NOT LINEAR.

1.9 : 2, 4, 8, 12, 18, 22, 26

(2.) $[T] = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

(4.) $[T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

MORE GENERALLY (AS IN EXAMPLE 3) IF $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ IS THE LINEAR TRANSFORMATION DEFINED BY ROTATION BY THE ANGLE θ (AS MEASURED COUNTERCLOCKWISE FROM THE POSITIVE x-AXIS) THEN

$[T_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

IN THIS CASE, $\theta = -\frac{\pi}{4}$.

(8.) $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(12.) $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix}$

SO $\theta = \frac{\pi}{2}$ (RADIANS) $= 90°$.

(18.) $T : \mathbb{R}^2 \to \mathbb{R}^4 : (x_1, x_2) \mapsto (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

$T(e_1) = T(1, 0) = (-3, 1, 0, 0)$

$T(e_2) = T(0, 1) = (2, -4, 0, 1)$

THUS $[T] = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(22.) $[T] = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$. $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. <u>CHECK</u>: $T(5,3) = (-1, 4, 9)$.

(26.) T IS <u>NOT</u> 1-1 BY THEOREM 12 BECAUSE THE COLUMNS OF $[T]$ ARE LINEARLY DEPENDENT.

T IS ONTO BY THEOREM 4 SINCE $[T]$ HAS A PIVOT POSITION IN BOTH ROWS:

$\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 \\ 0 & -19 & 19 \end{bmatrix}$.

2.1 : 2, 10

(2.) $A + 2B = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}$

$3C - E$ IS UNDEFINED BECAUSE $3C$ IS 2×2 AND E IS 2×1.

$CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$

EB IS UNDEFINED BECAUSE E IS 2×1 AND B IS 2×3 AND $1 \neq 2$.

(10.) $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$

~~$= AC$ YET $B \neq C$.~~

THIS PROBLEM ILLUSTRATES THE FACT THAT YOU CAN'T DIVIDE BY A MATRIX ; MATRIX DIVISION IS UNDEFINED.