MATH 22 LINEAR ALGEBRA FALL '04
HOMEWORK #4 ANSWER KEY

2.1 : 6, 22, 24, 32

(6.) (a.) $\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$, $\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$

THUS $\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$.

(b.) $\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4(1)+(-2)(2) & 4(3)+(-2)(-1) \\ (-3)(1)+0(2) & (-3)(3)+0(-1) \\ 3(1)+5(2) & 3(3)+5(-1) \end{bmatrix}$

$= \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$.

(22.) SUPPOSE A IS $m \times n$ AND B IS $n \times p$ SO THAT AB IS $m \times p$. WRITE $B = [v_1 \cdots v_p]$ WHERE $v_1, \ldots, v_p \in \mathbb{R}^n$. THE COLUMNS OF B ARE LINEARLY DEPENDENT, SO $c_1v_1 + \ldots + c_pv_p = \vec{0}$ FOR SCALARS $c_1, \ldots, c_p$ NOT ALL ZERO.
BY DEFINITION, $AB = [Av_1 \cdots Av_p]$.
NOW, $c_1(Av_1) + \ldots + c_p(Av_p)$
$= A(c_1v_1 + \ldots + c_pv_p) = A\vec{0} = \vec{0}$ AND THUS THE COLUMNS OF AB ARE LINEARLY DEPENDENT.

(24.) NOTICE THAT $x = Db$ IS A SOLUTION TO $Ax = b$ SINCE $A(Db) = (AD)b = I_m b = b$.
SINCE $AD = I_m$, A IS $m \times n$ AND D IS $n \times m$ FOR SOME $n \in \mathbb{N}$. THE COLUMNS OF $AD = I_m$ ARE LINEARLY INDEPENDENT, THUS THE COLUMNS OF D ARE LINEARLY INDEPENDENT BY THE CONTRAPOSITIVE OF EXERCISE 22.
THUS $m \leq n$. (ALSO, $n \leq m$ BECAUSE A HAS A PIVOT POSITION IN EVERY ROW.)

(32.) THIS IS INTUITIVELY CLEAR, BUT FOR A RIGOROUS PROOF, $AI_n = [Ae_1 \cdots Ae_n]$ BY DEFINITION. LET $A = [v_1 \cdots v_n]$ WHERE $v_1, \ldots, v_n \in \mathbb{R}^m$ ARE THE COLUMNS OF A. NOW, THE i^{TH} ENTRY OF e_j IS THE KRONECKER DELTA δ_i^j, DEFINED BY $\delta_i^j = \begin{cases} 1 & \text{IF } i=j \\ 0 & \text{IF } i \neq j. \end{cases}$

THUS $Ae_j = \sum_{i=1}^{n} \delta_i^j v_i = v_j, \quad j = 1, \ldots, n.$

THUS Ae_j IS JUST THE j^{TH} COLUMN OF A, $j=1,\ldots,n$ AND THEREFORE $AI_n = A$.

2.2: 4, 6, 14, 26

(4.) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ -\frac{7}{4} & \frac{3}{4} \end{bmatrix}$

(6.) $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$

(14.) $(B-C)D = 0 \Rightarrow BD - CD = 0 \Rightarrow BD = CD$
$\Rightarrow BDD^{-1} = CDD^{-1} \Rightarrow BI = CI.$
$B = BI$ AND $CI = C$, THUS $B = C$.

(26.) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

SIMILARLY, $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

2.2: 18, 24, 30, 32, 34, 38

(18.) $B = P^{-1}AP$.

PROOF: $A = PBP^{-1} \Rightarrow AP = PBP^{-1}P = PBI = PB$
$\Rightarrow P^{-1}AP = P^{-1}PB = IB = B$, THUS $B = P^{-1}AP$. QED

(24.) BY THEOREM 4, A HAS n PIVOT POSITIONS, THUS THE (UNIQUE) REDUCED ECHELON FORM OF A IS I_n. THUS A IS ROW EQUIVALENT TO I_n, SO $A = E_1 \cdots E_k I_n$ WHERE $E_1, \ldots, E_k$ ARE ELEMENTARY MATRICES. SINCE A IS A PRODUCT OF INVERTIBLE MATRICES, A IS INVERTIBLE. (LOOKING AHEAD, WE MAY USE THEOREM 8, P. 129.)

(30.) $$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \frac{1}{5} & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \frac{1}{5} & 0 \\ 0 & -1 & -\frac{4}{5} & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{4}{5} & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{7}{5} & 2 \\ 0 & 1 & \frac{4}{5} & -1 \end{bmatrix}$$

THUS THE INVERSE OF $\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$ IS $\begin{bmatrix} -\frac{7}{5} & 2 \\ \frac{4}{5} & -1 \end{bmatrix}$.

(32.) $$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}$$
NOT INVERTIBLE, BECAUSE NOT ROW EQUIVALENT TO I_3.

(34.) (a.) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$ (b.) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

(38.) (a.) $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b.) NO. BY WAY OF CONTRADICTION, SUPPOSE

$C \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = I_4$ FOR SOME 4×2 MATRIX C.

THE COLUMNS OF $C \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = I_4$ ARE

LINEARLY INDEPENDENT, THUS THE COLUMNS

OF $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ ARE LINEARLY INDEPENDENT

BY THE CONTRAPOSITIVE OF EXERCISE 2.1.22.

THIS IS A CONTRADICTION BECAUSE $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ARE LINEARLY DEPENDENT.

2.3: 4, 6, 14, 18, 28, 34, 38.

(4.) ZERO COLUMN $\Rightarrow$ LINEARLY DEPENDENT COLUMNS

$\Rightarrow$ NOT INVERTIBLE (BY THEOREM 8.)

(6.) $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

2 PIVOT POSITIONS $\Rightarrow$ NOT INVERTIBLE (BY THEOREM 8.)

(14.) A SQUARE TRIANGULAR MATRIX (UPPER OR LOWER) IS INVERTIBLE IFF ALL OF ITS DIAGONAL ENTRIES ARE NONZERO.

PROOF I: A MATRIX A IS INVERTIBLE IFF A^T IS INVERTIBLE, AND THE TRANSPOSE OF A LOWER TRIANGULAR MATRIX IS AN UPPER TRIANGULAR MATRIX WITH THE SAME DIAGONAL ENTRIES, SO IT SUFFICES TO PROVE THAT A SQUARE UPPER TRIANGULAR MATRIX IS INVERTIBLE IFF ALL OF ITS DIAGONAL ENTRIES ARE NONZERO. TO SEE THIS, SUPPOSE ALL DIAGONAL ENTRIES ARE NONZERO. THEN ALL DIAGONAL ENTRIES ARE PIVOT POSITIONS, AND THUS THERE IS A PIVOT POSITION IN EVERY ROW. CONVERSELY, IF ONE OR MORE DIAGONAL ENTRIES IS ZERO, THEN THERE ISN'T A PIVOT POSITION IN EVERY COLUMN, THUS THERE ISN'T A PIVOT POSITION IN EVERY ROW (THUS THE MATRIX IS NOT INVERTIBLE) QED

PROOF II: LOOKING AHEAD, THE DETERMINANT OF A SQUARE TRIANGULAR MATRIX IS THE PRODUCT OF ITS DIAGONAL ENTRIES, AND A SQUARE MATRIX IS INVERTIBLE IFF ITS DETERMINANT IS NONZERO. THUS A SQUARE TRIANGULAR MATRIX IS INVERTIBLE IFF THE PRODUCT OF ITS DIAGONAL ENTRIES IS NONZERO IFF ALL OF ITS DIAGONAL ENTRIES ARE NONZERO. QED

(18.) NO, BECAUSE C HAS FREE VARIABLES.

(28.) IF AB IS INVERTIBLE, SO IS B.

PROOF I: AB INVERTIBLE $\Rightarrow$ THERE EXISTS A MATRIX C SUCH THAT $CAB = I$, AND THUS CA IS THE INVERSE OF B. QED

PROOF II: BY CONTRAPOSITIVE. SUPPOSE B IS NOT INVERTIBLE. THEN B HAS LINEARLY DEPENDENT COLUMNS, AND THUS AB HAS LINEARLY DEPENDENT COLUMNS BY EXERCISE 2.1.22. THEREFORE AB IS NOT INVERTIBLE. QED

(34.) $[T] = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$

$[T]^{-1} = \begin{bmatrix} \frac{7}{2} & 4 \\ \frac{5}{2} & 3 \end{bmatrix}$

THUS $T^{-1}(x_1, x_2) = \left(\frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2\right)$.

(38.) $T(u) = T(v) \Rightarrow T(u) - T(v) = 0 \Rightarrow T(u-v) = 0$.
$u \neq v \Rightarrow u - v \neq 0$. THUS THERE IS A NONTRIVIAL SOLUTION TO THE HOMOGENEOUS SYSTEM $Tx = 0$ AND THEREFORE THE COLUMNS OF T ARE LINEARLY DEPENDENT, AND THUS T IS NOT SURJECTIVE (ONTO) BY THEOREM 8.

(ALSO, $T(u) = T(v)$ FOR $u \neq v$ MEANS T IS NOT INJECTIVE (1-1) AND THUS T HAS LINEARLY DEPENDENT COLUMNS.)

$T: \mathbb{R}^n \to \mathbb{R}^n$ IS ONTO IFF IT IS 1-1.