MATH 22 LINEAR ALGEBRA FALL '04

HOMEWORK #7 ANSWER KEY

4.4: 4, 8, 10, 14, 22, 26, 30, 32

(4.) $\begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & -7 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$

(8.) $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow [x]_\beta = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$

(10.) $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$

(14.) $\mathbb{P}_2$ IS ISOMORPHIC TO $\mathbb{R}^3$, SO WE NEED ONLY FIND THE COORDINATE VECTOR OF $\begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ RELATIVE TO THE BASIS $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}.$

$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$\Rightarrow [p(t)]_\beta = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$

(22.) LET $B = [b_1 \cdots b_n] \in M_n(\mathbb{R})$.

$A = B^{-1} = [b_1 \cdots b_n]^{-1} \in M_n(\mathbb{R})$.

IN OTHER WORDS, $A = P_\beta^{-1} \in M_n(\mathbb{R})$.

(26.) PROOF: $w = c_1 u_1 + \dots + c_p u_p \iff$

$$[w]_\beta = [c_1 u_1 + \dots + c_p u_p]_\beta \iff$$

$$[w]_\beta = P_\beta^{-1}[c_1 u_1 + \dots + c_p u_p] \iff$$

$$[w]_\beta = c_1 P_\beta^{-1}(u_1) + \dots + c_p P_\beta^{-1}(u_p) \iff$$

$$[w]_\beta = c_1 [u_1]_\beta + \dots + c_p [u_p]_\beta . \quad \text{QED}$$

(30.) $(1-t)^3 = 1 - 3t + 3t^2 - t^3$

$(2-3t)^2 = 4 - 12t + 9t^2$

SINCE $\mathbb{P}_3$ IS ISOMORPHIC TO $\mathbb{R}^4$, WE NEED ONLY TEST THE LINEAR DEPENDENCE OF

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -4 \end{bmatrix} \right\}. \qquad \text{LET } A = \begin{bmatrix} 1 & 4 & 0 \\ -3 & -12 & 0 \\ 3 & 9 & 3 \\ -1 & 0 & -4 \end{bmatrix}.$$

$\text{NUL } A = \text{SPAN} \left\{ \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \right\} \neq \{0\}$ AND THUS THE SET IS LINEARLY DEPENDENT. SPECIFICALLY,

$-4p_1 + p_2 + p_3 = 0.$

(32.) (a.) $\mathbb{P}_2$ IS ISOMORPHIC TO $\mathbb{R}^3$ SO WE NEED ONLY SHOW THAT $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \right\}$ IS A BASIS FOR $\mathbb{R}^3$. THIS IS THE CASE BECAUSE

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$$ IS INVERTIBLE.

(b.) $$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$$

SO $q(t) = 1 + 3t - 8t^2$.

4.5 : 2, 8, 12, 14, 22, 24, 26

(2.) (a.) A BASIS IS $\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(b.) THIS IS A TWO-DIMENSIONAL SUBSPACE OF $\mathbb{R}^3$, i.e. A PLANE THROUGH THE ORIGIN.

(8.) THIS IS THE NULLSPACE OF $\begin{bmatrix} 1 & -3 & 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3b-c \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a.) A BASIS IS $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b.) THIS IS A THREE-DIMENSIONAL SUBSPACE OF $\mathbb{R}^4$.

(12.) $$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 2 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

SO THE DIMENSION IS 3 BY THEOREM P. 260.

(14.) THE NULLITY (DIM NUL A) IS 3, AND THE RANK (DIM COL A) IS $6-3=3$ BY THEOREM 14, P. 265 (WHICH IS SOMETIMES CALLED THE FUNDAMENTAL THEOREM OF LINEAR ALGEBRA.)
(OR JUST USE THEOREM, P. 260.)

(22.) $\mathbb{P}_3$ IS ISOMORPHIC TO $\mathbb{R}^4$, SO WE NEED ONLY SHOW THAT $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix} \right\}$ IS A BASIS OF $\mathbb{R}^4$.

THIS IS THE CASE BECAUSE $\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

IS INVERTIBLE.

(24.) $\mathbb{P}_2$ IS ISOMORPHIC TO $\mathbb{R}^3$, SO WE NEED ONLY FIND THE COORDINATE VECTOR OF

$$\begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix} \text{ RELATIVE TO } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & -1 & -4 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

THUS $[p(t)]_\beta = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$.

(26.) PROOF: $\dim H = n \Rightarrow H$ HAS A BASIS CONSISTING OF n LINEARLY INDEPENDENT VECTORS IN H; AND THEREFORE H HAS A BASIS OF n LINEARLY INDEPENDENT VECTORS IN V, SINCE $H \subset V$. SINCE $\dim V = n$, n LINEARLY INDEPENDENT VECTORS IN V FORM A BASIS FOR V BY THE BASIS THEOREM. THEREFORE H AND V SHARE A BASIS, SO THEY ARE THE SAME VECTOR SPACE. QED

4.6 : 2, 6, 10, 14, 20, 24

(2.) RANK $A = 3$

DIM NUL $A = 2$

A BASIS FOR COL A IS: $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$.

A BASIS FOR ROW A IS:

$\{(1,-3,0,5,-7), (0,0,2,-3,8), (0,0,0,0,5)\}$.

$$x_1 = 3x_2 - 5x_4$$
$$x_3 = \tfrac{3}{2}x_4$$
$$x_5 = 0$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 \\ x_2 \\ \frac{3}{2}x_4 \\ x_4 \\ 0 \end{bmatrix}$$

SO A BASIS FOR NUL A IS $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$.

(6.) DIM NUL $A = 0$

DIM ROW $A = 3$

RANK $A^T = 3$

(10.) 1

(14.) IF A IS A 4×3 MATRIX OR A 3×4 MATRIX, THE LARGEST POSSIBLE DIMENSION OF ROW A IS 3.

MORE GENERALLY, IF A IS A $m \times n$ MATRIX, THE LARGEST POSSIBLE DIMENSION OF ROW A IS $\min\{m, n\}$ SINCE THIS IS THE MAXIMUM POSSIBLE NUMBER OF PIVOT POSITIONS.

(20.) 6×8 COEFFICIENT MATRIX A.
2 FREE VARIABLES $\Rightarrow$ DIM NUL $A = 2$
$\Rightarrow$ RANK $A = 6$. SINCE $\mathbb{R}^6$ IS THE ONLY 6-DIMENSIONAL SUBSPACE OF $\mathbb{R}^6$, COL $A = \mathbb{R}^6$.
THEREFORE, THE ANSWER IS NO.

(24.) 7×6 COEFFICIENT MATRIX A.
$Ax = b$ HAS A UNIQUE SOLUTION FOR SOME $b \in \mathbb{R}^7$ IFF THE 7×7 AUGMENTED MATRIX $[A \; b]$ HAS A PIVOT POSITION IN ALL COLUMNS BUT THE LAST. THIS IS POSSIBLE, SO THE ANSWER TO THE FIRST QUESTION IS YES.
HOWEVER, IT IS NOT POSSIBLE FOR THERE TO BE A UNIQUE SOLUTION FOR ALL $b \in \mathbb{R}^7$ BECAUSE IT IS NOT EVEN POSSIBLE FOR THERE TO BE A SOLUTION AT ALL FOR ALL $b \in \mathbb{R}^7$, SINCE THE MAXIMUM RANK OF A IS 6, SO THAT COL A IS A PROPER SUBSPACE OF $\mathbb{R}^7$. THEREFORE, THE ANSWER TO THE SECOND QUESTION IS NO.

5.1 : 2, 6, 8, 12, 18, 24, 32

(2.) YES, -2 IS AN EIGENVALUE OF $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

BECAUSE IT IS A ROOT OF THE CHARACTERISTIC POLYNOMIAL $\lambda^2 - 6\lambda - 16$.

ALTERNATIVELY,

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -2\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 7x + 3y = -2x \\ 3x - y = -2y \end{cases} \Rightarrow \begin{cases} 9x + 3y = 0 \\ 3x + y = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

SO -2 IS AN EIGENVALUE WITH EIGENSPACE

$$E_{-2}(A) = \text{SPAN}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}.$$

(6.) YES.

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

AND THE EIGENVALUE IS -2.

(8.) YES, 3 IS AN EIGENVALUE OF $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

BECAUSE $\det(A - 3I) = 0$.

ALTERNATIVELY, 3 IS AN EIGENVALUE BECAUS $\text{NUL}(A - 3I) = \text{SPAN}\left\{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right\}$

IS NONTRIVIAL. ANY NONZERO MULTIPLE OF $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ IS A CORRESPONDING EIGENVECTOR.

(12.) $\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} 7x+4y = x \\ -3x-y = y \end{cases}$

$\Rightarrow \begin{cases} 6x+4y=0 \\ -3x-2y=0 \end{cases} \Rightarrow y = -\frac{3}{2}x$

$\Rightarrow E_1(A) = \text{SPAN}\left\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right\}.$

$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix} \Rightarrow \begin{cases} 7x+4y = 5x \\ -3x-y = 5y \end{cases}$

$\Rightarrow \begin{cases} 2x+4y=0 \\ -3x-6y=0 \end{cases} \Rightarrow y = -\frac{x}{2}$

$\Rightarrow E_5(A) = \text{SPAN}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}.$

(18.) THE EIGENVALUES ARE 4, 0, −3 BY THEOREM 1.

(24.) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ HAS ONLY ONE DISTINCT EIGENVALUE, NAMELY 1, BY THEOREM 1.

(32) LET ℓ BE THE LINE THROUGH THE ORIGIN. T FIXES ℓ, SO 1 IS AN EIGENVALUE AND $E_1(T) = \ell$.