1. (36 points) Determine if the following statements are true or false. In each case, give either a short justification or example (as appropriate) to justify your conclusion.

T F (a) If A is a 4×4 matrix with characteristic polynomial

$$\lambda(\lambda - 1)(\lambda + 1)(\lambda + e),$$

then A is diagonalizable.

ANS: True. The matrix A has four distinct eigenvalues (0, 1, -1, and $-e$) and so is diagonalizable by Theorem 5.3.6.

T F (b) If A is invertible, then A is diagonalizable.

ANS: False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then the characteristic polynomial of A is

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2,$$

so the only eigenvalue of A is 1, with multiplicity 2. Since

$$A - I_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

all the eigenvectors of A are multiples of $(0, 1)$; i.e., the 1-eigenspace of A is 1-dimensional. Thus, Theorem 5.3.7 implies that A is *not* diagonalizable.

T F (c) If A is a symmetric matrix such that $A^3 = 0$, then $A = 0$.

ANS: True. Suppose $A\mathbf{x} = \lambda\mathbf{x}$ for some non-zero $\mathbf{x}$. Then

$$0 = A^3\mathbf{x} = A^2(\lambda\mathbf{x}) = \lambda(A^2\mathbf{x}) = \lambda^2(A\mathbf{x}) = \lambda^3\mathbf{x}$$

implies $\lambda^3 = 0$; hence, the *only* eigenvalue of A is 0. But A symmetric means that A is orthogonally diagonalizable by the Spectral Theorem for Symmetric Matrices. In particular, A is diagonalizable. Then the Diagonalization Theorem implies that $A = PDP^{-1}$ for some diagonal matrix D and some orthogonal matrix P. Since the entries on the main diagonal of D are eigenvalues of A, $D = 0$, and $A = 0$.

T F (d) If A is an $m \times n$ matrix with orthonormal columns, then $n \leq m$.

ANS: True. Recall that since the columns of A are orthonormal, each of the columns of A must be non-zero. Thus, the columns of A form an orthogonal set of non-zero vectors in $\mathbb{R}^m$. By Theorem 6.2.4, this means that the columns of A form a linearly independent set, so A cannot have more columns than rows by Theorem 1.7.8; i.e., $n \leq m$.

T F (e) If A is an $n \times n$ symmetric matrix, then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$.

ANS: True. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then by the definition of the dot product, Theorem 2.1.3(d), the fact that $A^T = A$ (since A is symmetric), and the associativity of matrix multiplication,

$$(A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(A^T\mathbf{y}) = \mathbf{x}^T(A\mathbf{y}) = \mathbf{x} \cdot (A\mathbf{y}).$$

(Problem 1 continued on next page)

T **F** (f) If A and B are invertible $n \times n$ matrices, then so is $A+B$.

ANS: False. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$
Since $\det A = 1 = \det B$, A and B are invertible by the Invertible Matrix Theorem. But
$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
and $\det(A+B) = 0$, so $A+B$ is *not* invertible by the Invertible Matrix Theorem.

T **F** (g) If A is a 3×5 matrix and $\dim \operatorname{Nul} A = 2$, then $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^3$.

ANS: True. By the definition of rank A and the Rank Theorem,
$$\dim \operatorname{Col} A = \operatorname{rank} A = 5 - \dim \operatorname{Nul} A = 5 - 2 = 3;$$
hence, Col A is a 3-dimensional subspace of $\mathbb{R}^3$, so Col $A = \mathbb{R}^3$. The box following Theorem 4.2.3 thus implies that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^3$.

T **F** (h) The set of $n \times n$ matrices A satisfying $\det A = 0$ is a subspace of $M_{n \times n}$, the set of $n \times n$ matrices.

ANS: False. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
and note that $\det A = 0 = \det B$. But
$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
has determinant 1; i.e., the set of $n \times n$ matrices A satisfying $\det A = 0$ is *not* closed under addition.

T **F** (i) If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^2$ and $\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = 10$, then the area of the triangle in the plane with vertices $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ is 10.

ANS: False. Let
$$\mathbf{u} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
Then
$$\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{vmatrix} 10 & 0 \\ 0 & 1 \end{vmatrix} = 10,$$
but the triangle with vertices $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ is a right triangle with legs of length 1 and 10 and hence has area 5.

2. (15 points) Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of V such that $\mathcal{C} = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.

(a) Show that $\mathcal{B}$ must be a linearly independent set.

ANS: Suppose $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Then $T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = T(\mathbf{0})$. Since T is a linear transformation, this gives $c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n) = \mathbf{0}$, and $\mathcal{C}$ a linearly independent set implies $c_1 = \cdots = c_n = 0$. It follows that $\mathcal{B}$ must be a linearly independent set by definition.

(b) Must T map V onto W?

ANS: Yes. First note that since V is a vector space and $\mathcal{B}$ is a subset of V, any linear combination of the vectors in $\mathcal{B}$ is in V. Since $\mathcal{C}$ is a basis for W, any $\mathbf{x} \in W$ has the form $\mathbf{x} = c_1T(\mathbf{v}_1)+\cdots+c_nT(\mathbf{v}_n)$ for some $c_1, \ldots, c_n \in \mathbb{R}$. But T a linear transformation implies $\mathbf{x} = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n)$.

(c) Must T be one-to-one?

ANS: No. Let $T : \mathbb{R}^2 \to \mathbb{R}$ be given by $T(x, y) = x$. Let $\mathbf{v}_1 = (1, 0)$ and $\mathcal{B} = \{\mathbf{v}_1\}$. Note that $\mathcal{C} = \{T(\mathbf{v}_1)\} = \{1\}$ is a basis for $\mathbb{R}$. But $T(0, 0) = 0 = T(0, 1)$, so T is *not* one-to-one by definition.

3. (20 points) Let

$$A = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}.$$

Note that the characteristic polynomial of A is

$$-\lambda(\lambda - 9)^2.$$

(a) Find an *orthogonal* basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

ANS: Note that by the Diagonalization Theorem, $\mathbb{R}^3$ *does* have a basis consisting of eigenvectors of A since A is symmetric and hence is orthogonally diagonalizable (by the Spectral Theorem for Symmetric Matrices) and so is diagonalizable. In addition, the eigenvalues of A are 0, 9, and 9. Since the sum of each row of A is 0, an eigenvector corresponding to the eigenvalue 0 is $(1, 1, 1)$, so it remains to find eigenvectors corresponding to the eigenvalue 9. But

$$A - 9I_3 = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

implies that $(-1, 1, 0)$ and $(-1, 0, 1)$ are linearly independent eigenvectors corresponding to the eigenvalue 9, and

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is an eigenvector basis for $\mathbb{R}^3$. To make this basis *orthogonal*, we use the Gram-Schmidt Process:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 0\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\mathbf{v}_2 - 0\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

(Problem 3 continued on next page)

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

ANS: The matrix D is

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

To find the matrix P, we need to make the orthogonal basis we found in the last part an *orthonormal* basis. Since

$$\|(1,1,1)\| = \sqrt{3}, \qquad \|(-1,1,0)\| = \sqrt{2}, \qquad \text{and} \qquad \|(-1/2,-1/2,1)\| = \sqrt{\frac{3}{2}},$$

we have

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \end{bmatrix}.$$

(c) Find a symmetric matrix B such that $B^2 = A$.

ANS: Recall that since $A = PDP^T$, where P and D are as in the last part, $A^2 = PD^2P^T$. Thus, if

$$B = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \end{bmatrix}^T,$$

then B is symmetric and $B^2 = A$.

4. (9 points) Let -1 and 1 be eigenvalues of a matrix A. Suppose u_1 and u_2 are linearly independent eigenvectors of A corresponding to -1, and suppose w_1 and w_2 are linearly independent eigenvectors of A corresponding to 1. Show that $\{u_1, u_2, w_1, w_2\}$ is a linearly independent set.

ANS: Suppose $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $c_1u_1 + c_2u_2 + c_3w_1 + c_4w_2 = 0$. Then $c_1u_1 + c_2u_2 = -c_3w_1 - c_4w_2$ is in the eigenspace of A corresponding to -1 and the eigenspace of A corresponding to 1. Since a non-zero vector cannot correspond to two distinct eigenvalues of A, this implies that $c_1u_1 + c_2u_2 = -c_3w_1 - c_4w_2 = 0$. Then $\{u_1, u_2\}$ and $\{w_1, w_2\}$ linearly independent sets imply c_1, c_2, c_3, c_4 are all zero, and by definition, $\{u_1, u_2, w_1, w_2\}$ is a linearly independent set.

5. (15 points) Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be the standard basis for $M_{2\times 2}$, the set of 2×2 matrices. Find the $\mathcal{B}$-matrix for the linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(A) = A^T$.

ANS: Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and note that

$$[A_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [A_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [A_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [A_4]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $T(A_1) = A_1$, $T(A_2) = A_3$, $T(A_3) = A_2$, and $T(A_4) = A_4$,

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by definition.

6. (15 points) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be a linear transformation, and let

$$\mathcal{B} = \{1, t, t^2\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$$

be bases for $\mathbb{P}_2$ and $\mathbb{R}^2$, respectively. Suppose the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

(a) Find $[-3 - t + 2t^2]_{\mathcal{B}}$.

ANS: By definition,

$$[-3 - t + 2t^2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}.$$

(b) Find $[T(-3 - t + 2t^2)]_{\mathcal{C}}$.

ANS: By Equation (3) on page 328, $[T(-3 - t + 2t^2)]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}}[-3 - t + 2t^2]_{\mathcal{B}}$, so

$$[T(-3 - t + 2t^2)]_{\mathcal{C}} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}.$$

(c) Find $T(-3 - t + 2t^2)$.

ANS: Since

$$[T(-3 - t + 2t^2)]_{\mathcal{C}} = \begin{bmatrix} -2 \\ -7 \end{bmatrix},$$

we have

$$T(-3 - t + 2t^2) = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -23 \end{bmatrix}.$$

7. (10 points) For which real numbers a, b, c is the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 17 & 0 \\ b & c & 1 \end{bmatrix}$$

diagonalizable?

ANS: Note that the eigenvalues of A are 1, 1, and 17 by Theorem 5.1.1. Thus, Theorem 5.3.7 implies that for A to be diagonalizable, the 1-eigenspace must be 2-dimensional. We must thus find values for a, b, c such that the equation $(A - I_3)\mathbf{x} = \mathbf{0}$ has two free variables. Since

$$A - I_3 = \begin{bmatrix} 0 & 0 & 0 \\ a & 16 & 0 \\ b & c & 0 \end{bmatrix},$$

we need $(b, c) = r(a, 16)$ for some $r \in \mathbb{R}$. If $a \neq 0$, then $b = ar$ implies $r = b/a$; hence, $c = 16r = 16b/a$. If $a = 0$, then $b = ar$ and $c = 16r$ imply $b = 0$ and c is any real number. It follows that the matrix A is diagonalizable when either

$$0 \neq a \in \mathbb{R}, \qquad b \in \mathbb{R}, \qquad \text{and} \qquad c = \frac{16b}{a}$$

or

$$a = b = 0 \qquad \text{and} \qquad c \in \mathbb{R}.$$

8. (15 points) Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix},$$

and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}_1$.

ANS: By definition, the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}_1$ is

$$\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = -\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Find $\text{proj}_W \mathbf{y}$.

ANS: Note that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, so by the Orthogonal Decomposition Theorem,

$$\text{proj}_W \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Find the distance from $\mathbf{y}$ to W.

ANS: By the Best Approximation Theorem, the distance from $\mathbf{y}$ to W is

$$\|\mathbf{y} - \text{proj}_W \mathbf{y}\| = \|(-2, 1, 0, 3) - (-1, 0, 1, 0)\| = \|(-1, 1, -1, 3)\| = \sqrt{12}.$$

9. (15 points) Show that if $\mathbf{v}$ is an eigenvector of an $n \times n$ matrix A and $\mathbf{v}$ corresponds to a non-zero eigenvalue of A, then $\mathbf{v} \in \text{Col } A$.

ANS: Let $0 \neq \lambda \in \mathbb{R}$ such that $\mathbf{v}$ is an eigenvector of A corresponding to λ. Then $A\mathbf{v} = \lambda\mathbf{v}$ by definition, and $\lambda \neq 0$ implies $\mathbf{v} = \lambda^{-1}(A\mathbf{v}) = A(\lambda^{-1}\mathbf{v})$; i.e., the equation $A\mathbf{x} = \mathbf{v}$ is consistent. It follows from the box following Theorem 4.2.3 that $\mathbf{v} \in \text{Col } A$.

Name: ____________________

Math 22

2006-12-03

Final Exam

Instructions: You have three hours to complete this exam. Show **all** your work on the exam itself. If you need additional space, write your name clearly at the top of any additional sheets, and submit them together with the exam.

You may not use calculators or computers. You must work alone and neither receive nor provide assistance to anyone else.

Problem	Points	Score
1	36	
2	15	
3	20	
4	9	
5	15	
6	15	
7	10	
8	15	
9	15	
Total	150	