Workshop II

2006-09-28

1. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^n$ is *linearly independent* if whenever $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{0}$ for $c_1, c_2, c_3 \in \mathbb{R}$, $c_1 = c_2 = c_3 = 0$; otherwise, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is *linearly dependent*. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$, and let

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3$$

with $d_1, d_2, d_3 \in \mathbb{R}$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}\}$ is linearly dependent.

2. Let A be an $m \times n$ matrix, and let $\mathbf{b}, \mathbf{b}' \in \mathbb{R}^m$ and $c \in \mathbb{R}$.

   (a) Prove that if the equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{b}'$ are both consistent, then the equation $A\mathbf{x} = \mathbf{b} + \mathbf{b}'$ is consistent.

   (b) Prove that if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then the equation $A\mathbf{x} = c\mathbf{b}$ is consistent.

3. Let A be an $m \times n$ matrix, and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

   (a) Prove that if $\mathbf{u}$ and $\mathbf{v}$ are solutions to the equation $A\mathbf{x} = \mathbf{0}$, then so is $\mathbf{u} + \mathbf{v}$.

   (b) Prove that if $\mathbf{u}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$, then so is $c\mathbf{u}$.

   (c) Is it true that $\mathbf{u}$ and $\mathbf{v}$ are solutions to the equation $A\mathbf{x} = \mathbf{0}$ *if and only if* $\mathbf{u} + \mathbf{v}$ is?

   (d) If $c \neq 0$, is it true that $\mathbf{u}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$ *if and only if* $c\mathbf{u}$ is?

4. Let A be an $m \times n$ matrix. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are all solutions to the equation $A\mathbf{x} = \mathbf{0}$ and $\mathbf{v} \in \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$. Show that $\mathbf{v}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$.

5. For each of the following statements, decide whether it is true or false. If it is true, prove it; if it is false, give a counterexample.

   (a) If the vectors $\mathbf{u}$ and $\mathbf{v}$ are solutions to the equation $A\mathbf{x} = \mathbf{b}$, then so is $\mathbf{u} + \mathbf{v}$.

   (b) Let A and B be 2×2 matrices, and let $\mathbf{u} \in \mathbb{R}^2$. Then $A(B\mathbf{u}) = B(A\mathbf{u})$.

6. Prove that if n is an integer such that n^2 is even, then n is even.