Row reduction algorithm

Solve the following system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ x_1 + 4x_2 + 3x_3 &= 12 \\ 2x_1 \qquad\quad + x_3 &= 3 \end{aligned}$$

Solution: Write down the augmented matrix. Reduce it first to echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 1 & 4 & 3 & 12 \\ 2 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 2 & 0 & 4 \\ 0 & -4 & -5 & -13 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -4 & -5 & -13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -5 & -5 \end{array}\right]$$

Continue to reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 1$.

Transform the following matrix first into echelon, and then reduced echelon form.

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

This matrix is in echelon form.

Create reduced echelon form:

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrrr|r} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

- Mark pivot columns.

- Basic variable: Any variable that correspond to a pivot column.
- Free variable: All nonbasic variables.

What is the solution(s) to this system?

Solve the following system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3 \\ x_1 + 4x_2 + 4x_3 &= 5 \\ 2x_2 + x_3 &= 3 \end{aligned}$$

Solution: Write down the augmented matrix. Reduce it first to echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 1 & 4 & 4 & 5 \\ 0 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent, it has no solution.
No need to proceed further.

Row reduction algorithm

Solve the following system:

$$\begin{array}{rrrrl} x_1 + & 2x_2 + & 3x_3 + & 2x_4 = & 14 \\ x_1 + & 4x_2 + & 3x_3 + & 2x_4 = & 18 \\ 2x_1 & & + x_3 + & 3x_4 = & 12 \\ 3x_1 + & x_2 & & + 2x_4 = & 11 \end{array}$$

Solution: Write down the augmented matrix. Reduce it first to echelon form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 1 & 4 & 3 & 2 & 18 \\ 2 & 0 & 1 & 3 & 12 \\ 3 & 1 & 0 & 2 & 11 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & -4 & -5 & -1 & -16 \\ 0 & -5 & -9 & -4 & -31 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -5 & -1 & -8 \\ 0 & 0 & -9 & -4 & -21 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1/5 & 8/5 \\ 0 & 0 & 9 & 4 & 21 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1/5 & 8/5 \\ 0 & 0 & 0 & 11/5 & 33/5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1/5 & 8/5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Continuing....

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 14 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1/5 & 8/5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 8 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

The solution is: $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.