$$\frac{87+6}{87} = \frac{93}{87} \ !$$ amazing - congratulations!

Math 22: Linear Algebra FINAL, Summer 2006

No calculators; one letter sheet of notes allowed. Your NAME: Vissuta Jiwariyavej

1. [12 points] Consider the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $A = \begin{bmatrix} 1/2 & 0 \\ 5/2 & -2 \end{bmatrix}$

   (a) Categorize the stability of the origin: is it a repellor, attractor, or saddle point?

   Find eigenvalue $\det(A-\lambda I) = \det\begin{bmatrix} 1/2-\lambda & 0 \\ 5/2 & -2-\lambda \end{bmatrix} = (1/2-\lambda)(-2-\lambda) \Rightarrow \lambda = 1/2, -2$

   $|-2|>1$ and $|1/2|<1 \Rightarrow$ so the origin is the saddle point.

   (b) Given the initial condition $\mathbf{x}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, compute $\mathbf{x}_1$ and $\mathbf{x}_5$. [Hint: you will not want to do this via simple matrix multiplication!]

   For $\lambda = 1/2$ $\quad A-\lambda I = \begin{bmatrix} 1/2-1/2 & 0 \\ 5/2 & -2-1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 5/2 & -5/2 \end{bmatrix}$ eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $\lambda = -2$ $\quad A+2I = \begin{bmatrix} 1/2 & 0 \\ 5/2 & -2+2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 5/2 & 0 \end{bmatrix}$ eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

   $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{v}_1\ \vec{v}_2]^{-1}\vec{x}_0$

   $= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   $= \frac{1}{1\cdot 1 - 1\cdot 0}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

   $\vec{x}_0 = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$

   $x_1 = 2\cdot\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3(-2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

   $x_2 = 2\cdot(\frac{1}{2})^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3(-2)^2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ -12 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -23/2 \end{bmatrix}$

   $x_3 = 2\cdot(\frac{1}{2})^3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3(-2)^3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} + \begin{bmatrix} 0 \\ 24 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 97/4 \end{bmatrix}$

   $x_4 = 2\cdot(\frac{1}{2})^4\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3(-2)^4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 1/8 \end{bmatrix} + \begin{bmatrix} 0 \\ -48 \end{bmatrix} = \begin{bmatrix} 1/8 \\ -47\frac{7}{8} \end{bmatrix}$

   $x_5 = 2\cdot(\frac{1}{2})^5\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3(-2)^5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/16 \\ 1/16 \end{bmatrix} + \begin{bmatrix} 0 \\ 96 \end{bmatrix} = \begin{bmatrix} 1/16 \\ 96\frac{1}{16} \end{bmatrix}$

   (c) Find a formula (in terms of numbers, and the symbol k, only) for the entries of $\mathbf{x}_k$, for general $k \geq 1$.

   $\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$

   $= 2(\frac{1}{2})^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-3)(-2)^k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   $= 2^{1-k}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ (-3)(-2)^k \end{bmatrix}$

   $= \begin{bmatrix} 2^{1-k} + 0 \\ 2^{1-k} + (-3)(-2)^k \end{bmatrix} = \begin{bmatrix} 2^{1-k} \\ 2^{1-k} - 3\cdot(-2)^k \end{bmatrix}$

   (d) To what direction will $\mathbf{x}_k$ tend as $k \to \infty$? (You may give any nonzero vector in this direction).

   ~~$\vec{v}_1\ (\lambda = \frac{1}{2})$~~

   It will swink back and forth but converge to $\vec{v}_2$ $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

12/12

2. [11 points] An engineering problem gives the linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ a \end{bmatrix}$, where a can vary.

(a) For what values of a is the system consistent?

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & a \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -5 \\ 0 & -1 & a-3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & a+2 \end{array}\right]$$

for the system to be consistent : $a+2 = 0$

(otherwise will have row of $[0\ 0 \cdots \text{constant}]$) $a = -2$.

(b) When consistent, is the solution unique? Why?

The solution is unique because there is no free variable.

(A has 2 pivots for $A = 2\times 3$.)

(c) When $a = 0$, find the value of $\mathbf{x}$ which gives the least-squares solution to the system.

$\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ let $\vec{v}_1, \vec{v}_2$ be the first and second column of A respectively.

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{6\cdot 2 - 3\cdot 3} \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 3 \end{bmatrix}$$

(d) In this case, what is the minimum approximation error $\|A\mathbf{x} - \mathbf{b}\|$?

$$A\hat{x} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2/3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/3 + 3 \\ -4/3 + 3 \\ -2/3 + 0 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5/3 \\ -2/3 \end{bmatrix}$$

minimum error $= \| A\hat{x} - \vec{b} \| = \left\| \begin{bmatrix} 7/3 \\ 5/3 \\ -2/3 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2/3 \\ 2/3 \\ -2/3 \end{bmatrix} \right\| = \sqrt{(-2/3)^2 + (2/3)^2 + (-2/3)^2}$

$= \sqrt{4/9 + 4/9 + 4/9}$

$= \sqrt{12}/3 = 2/\sqrt{3}$

11/11

3. [13 points] In parts a–c you don't necessarily need to diagonalize the matrix in order to answer the question. Also at least one eigenvalue will be obvious each time.

(a) Is the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ diagonalizable? Why?

A is diagonalizable because it is a triangular matrix with 3 different entries on the diagonal, that means A has 3 distinct eigenvectors. (L.I.)

(b) Is the matrix $B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ diagonalizable? Why?

$B - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix} \Rightarrow \det(B-\lambda I) = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$= (2-\lambda)((3-\lambda)^2 - 1)$

$= (2-\lambda)(9-6\lambda+\lambda^2-1)$

$= (2-\lambda)(8-6\lambda+\lambda^2)$

$= (2-\lambda)(4-\lambda)(2-\lambda)$

$\lambda = 2, 2, 4$

For $\lambda = 2$, $B - \lambda I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = x_1 \\ x_2 = 0 \\ x_3 = 0 \end{array} \Rightarrow \text{eigenvector} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

1 dimensional.

but we need 2 L.I. eigenvectors for $\lambda = 2$.

So B is not diagonalizable

(c) Is the matrix $C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ diagonalizable? Why?

$\det(C-\lambda I) = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$ (equals to $\det(B-\lambda I)$)

So $\lambda = 2, 2, 4$

For $\lambda = 2$ $C - \lambda I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{array}{l} x_1 = x_1 \\ x_2 = -x_3 \\ x_3 = x_3 \end{array}$

$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Nul $(C - \lambda I)$ is 2-dimensional.

So we have 3 L.I. eigenvectors (2 from $\lambda = 2$, 1 from $\lambda = 4$)

So C is diagonalizable

(d) Take C to be whichever of the last two above matrices b) or c) was diagonalizable. Find a matrix P and a diagonal matrix D such that $C = PDP^{-1}$. [Note: several points available here, so take care].

C is from c) $\quad C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$\lambda = 2, 2, 4$

for $\lambda = 2$ eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ (from previous part)

when $\lambda = 4$ $\quad C - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \underset{R_3 \to R_3 + R_2}{\sim} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 \to R_1 + R_2}{\sim} \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2 \to -R_2}{\overset{R_1 \to -\frac{1}{2}R_1}{\sim}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = x_3$
$x_2 = x_3 \Rightarrow$ eigenvector $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$x_3 = x_3$

matrix of eigenvectors $\downarrow$

matrix of eigenvalues. $\downarrow$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

13/13

4. [11 points]

(a) $\mathcal{B} = \{1, 1+2t, -1+t^2\}$ is a basis for $\mathbb{P}_2$, the vector space of all degree-2 polynomials. Find the $\mathcal{B}$-coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ of the polynomial $\mathbf{p}(t) = 1 + 2t + 3t^2$.

$\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$

So translate B to $\mathbb{R}^3 \Rightarrow \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$p(t) \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$A[\vec{x}]_B = \vec{x} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2 \leftarrow \frac{1}{2}R_2]{R_1 \leftarrow R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{R_1 \leftarrow R_1 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$[\vec{p}]_{\mathcal{B}} = [\vec{x}]_B = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

check $3(1) + 1(1+2t) + 3(-1+t^2) = 3 + 1 + 2t - 3 + 3t^2 = 1 + 2t + 3t^2$

(b) Let U and V be orthogonal matrices. Prove that UV is then also an orthogonal matrix.

Since U and V are orthogonal matrix,

$U^TU = I$ and $V^TV = I$

$V^TV = I$

$V^TIV = I$

$V^TU^TUV = I$

$(UV)^TUV = I$

Therefore UV is also an orthogonal matrix

(W orthogonal matrix $\Leftrightarrow W^TW = I$)

(c) If A and B are $n \times n$ matrices with the same set of n linearly independent eigenvectors, but possibly different eigenvalues, prove that A and B commute.

So $\left.\begin{array}{l} A = PDP^{-1} \\ B = PEP^{-1} \end{array}\right\}$ P = matrix of eigenvectors

D = matrix of eigenvalues of $A = \begin{bmatrix} \lambda_{A1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{An} \end{bmatrix}$

E = matrix of eigenvalues of $B = \begin{bmatrix} \lambda_{B1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{Bn} \end{bmatrix}$

$AB = PDP^{-1}PEP^{-1}$

$= PDEP^{-1}$

$BA = PEP^{-1}PDP^{-1}$

$= PEDP^{-1}$

But $DE = \begin{bmatrix} \lambda_{A1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{An} \end{bmatrix} \begin{bmatrix} \lambda_{B1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{Bn} \end{bmatrix} = \begin{bmatrix} \lambda_{A1}\lambda_{B1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{An}\lambda_{Bn} \end{bmatrix}$

$ED = \begin{bmatrix} \lambda_{B1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{Bn} \end{bmatrix} \begin{bmatrix} \lambda_{A1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{An} \end{bmatrix} = \begin{bmatrix} \lambda_{A1}\lambda_{B1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{An}\lambda_{Bn} \end{bmatrix}$

because all entries are on diagonal the multiplication is super easy.

So $DE = ED$

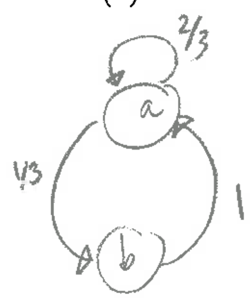
$PDEP^{-1} = PEDP^{-1}$

Therefore $AB = BA \Rightarrow A$ and B commute

11/11

5. [8 points] Assume all stock prices can have only two possible values, \$1 (state a) and \$2 (state b). Let $\mathbf{x}_k$ represent the probability vector whose two entries give the fraction of stocks in state a and in state b respectively. The evolution of the market each day is given by the following. Of the stocks in state a, one third of them jump up to b and the other two thirds stay in a. Of the stocks in state b, all of them jump down to a.

(a) Construct the stochastic matrix for the Markov chain.



$$\text{Stochastic matrix } A = \begin{array}{c|cc} & \text{from } a & b \\ \text{to } a & 2/3 & 1 \\ b & 1/3 & 0 \end{array} \qquad A=\begin{bmatrix} 2/3 & 1 \\ 1/3 & 0 \end{bmatrix}$$

2

$$A^2 = \begin{bmatrix} 2/3 & 1 \\ 1/3 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & 1 \\ 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 7/9 & 2/3 \\ 2/9 & 1/3 \end{bmatrix}$$ all entries >0

So A is regular

(b) Find the steady state probability vector.

A is regular so there's a unique steady state $\vec{q}$ that $A\vec{q} = \vec{q}$

So $\vec{q}$ is and eigenvector for $\lambda = 1$

when $\lambda = 1$ $\quad A - \lambda I = \begin{bmatrix} -1/3 & 1 \\ 1/3 & -1 \end{bmatrix} \Rightarrow$ eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow$ normalize to $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$

So the steady state probability vector is $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$

(c) Does *every* starting probability vector $\mathbf{x}_0$ converge to this steady state vector? (Why?) yes Because A is regular and therefore the system has a unique steady state.

(d) [n-point BONUS]: If $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *estimate* how many days it will take to 'settle' so that each entry in $\mathbf{x}$ is within 1% of its steady state value.

Diagonalize A

$$A - \lambda I = \begin{bmatrix} 2/3-\lambda & 1 \\ 1/3 & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (2/3-\lambda)(-\lambda) - \tfrac{1}{3}\cdot 1$$
$$0 = -\tfrac{2}{3}\lambda + \lambda^2 - \tfrac{1}{3}$$
$$0 = 3\lambda^2 - 2\lambda - 1$$
$$0 = (3\lambda+1)(\lambda-1)$$
$$\lambda = -1/3,\ 1$$

for $\lambda = -1/3$

$$A - \lambda I = \begin{bmatrix} 2/3+1/3 & 1 \\ 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/3 & 1/3 \end{bmatrix} \quad \text{eigenvector} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} -1/3 & 1 \\ 1/3 & -1 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \quad P^{-1} = \frac{1}{1\cdot 1+3\cdot 1}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-\frac{1}{3})^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{bmatrix}$$

$$\vec{x}_k = A^k\vec{x}_0 = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-\frac{1}{3})^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & -3/4 \\ 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-\frac{1}{3})^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{3}{4}(-\frac{1}{3})^k \\ 1/4 \end{bmatrix}$$

$$\vec{x}_k = \begin{bmatrix} -\frac{3}{4}(-\frac{1}{3})^k + \frac{3}{4} \\ \frac{3}{4}(-\frac{1}{3})^k + \frac{1}{4} \end{bmatrix}$$

continue

$\frac{8+3}{8}$

$$\vec{x}_k = \begin{bmatrix} -\frac{3}{4}\left(-\frac{1}{3}\right)^k + \frac{3}{4} \\ \frac{3}{4}\left(-\frac{1}{3}\right)^k + \frac{1}{4} \end{bmatrix} \qquad \text{steady state} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$$

for $\vec{x}_k$ to be within 1% of steady state

(1) $\left| -\frac{3}{4}\left(-\frac{1}{3}\right)^k \right| \leqslant \frac{1}{100} \cdot \frac{3}{4}$ and (2) $\left| \frac{3}{4}\left(-\frac{1}{3}\right)^k \right| \leq \frac{1}{100} \cdot \frac{1}{4}$

So it's enough to find k that satisfies (2).

good

$$\left| \frac{3}{4}\left(-\frac{1}{3}\right)^k \right| \leqslant \frac{1}{400}$$

$$\left| \left(-\frac{1}{3}\right)^k \right| \leq \frac{1}{300}$$

$$\left| \left(\frac{1}{3}\right)^k \right| \leqslant \frac{1}{300}$$

$$3^k \geqslant 300$$

$3^3 = 27$

$3^5 = 243$

$3^6 = 729$

So $k \geqslant 6$

(assuming k is an integer.)

So it'll take 6 days to settle within 1% of steady state.

excellent.

+3

6. [12 points] The matrix A has been converted to reduced echelon form as follows

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -2 & -4 & 3 & -5 \\ 2 & 4 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for Row A

Basis for Row A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ because row reduction preserves span of Row space.

(b) Find a basis for Col A

Basis for Col A is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right\}$ because row reduction doesn't preserve span of Column Space But the pivot columns in the original matrix are basis for Col A.

(c) What is rank A?

$\text{rank} A = \dim \text{Col} A = \dim \text{Row} A = 2$

(d) Find a basis for $(\text{Col } A)^{\perp}$. [Hint: to make this easier write Col A as the span of the set from b), then write this as the column or row space of a new (smaller) matrix]. $(\text{Col} A)^{\perp} = \text{Nul}(A^T)$

$$A^T = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 0 & 3 & -1 \\ 1 & -5 & 3 \end{bmatrix} \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_4 \leftarrow R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -3 & 1 \end{bmatrix} R_4 \leftarrow R_4 + R_3 \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \leftarrow \tfrac{1}{3} R_3 \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} R_1 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -4/3 x_3 \\ x_2 &= 1/3 x_3 \\ x_3 &= x_3 \end{aligned} \Rightarrow \vec{x} = x_3 \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}$$

So basis of $\text{Nul}(A^T)$ which is also basis of $(\text{Col} A)^{\perp}$ is $\begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}$

(e) The lines Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ are subspaces of $\mathbb{R}^3$ and meet at right-angles. Are they each other's orthogonal complement? Explain.

No. Both Spans are one-dimensional. But in order to be each other's orthogonal complement, the combination of the two spans needs to span $\mathbb{R}^3$ (or the whole space), which is 3-dimensional. The combination of them are only 2-dimensional. So they're not each other's orthogonal complement.

12/12

7. [10 points]

(a) Compute the determinant of the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 5 & 2 & 3 \\ 1 & -2 & 2 & 6 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ [Hint: you may want to mix two techniques].

$$\det A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 5 & 2 & 3 \\ 1 & -2 & 2 & 6 \\ 1 & 3 & 4 & 5 \end{vmatrix} = 0 - 1\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 6 \\ 1 & 4 & 5 \end{vmatrix} + 0 - 0$$

$$= -\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 6 \\ 1 & 4 & 5 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 2 & 2 \end{vmatrix}$$

$$= -\left(1\begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} - 0 + 0\right)$$

$$= -\begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} = -(0 \cdot 2 - 2 \cdot 3) = 6$$

(b) Use your above result to determine whether or not the matrix has a zero eigenvalue, or state that this cannot be determined using this result.

$\det(A - 0I) = \det A = 6$. Therefore A does not have a zero eigenvalue.

(c) Find the orthogonal projection of the point $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$ onto the plane $W =$ Span $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}} \right\}$. [Hint: use some nice property of the spanning set].

$\vec{v}_1 \cdot \vec{v}_2 = 1 \cdot 1 + 1 \cdot (-1) + 0 \cdot 2 = 0 \Rightarrow$ they are orthogonal basis.

So the projection on W of $\vec{x}$ is $\hat{x} = \dfrac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \dfrac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$

$$= \frac{2 \cdot 1 + 0 \cdot 1 + (-7) \cdot 0}{1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0}\vec{v}_1 + \frac{2 \cdot 1 + 0(-1) + (-7) \cdot 2}{1 \cdot 1 + (-1)(-1) + 2 \cdot 2}\vec{v}_2$$

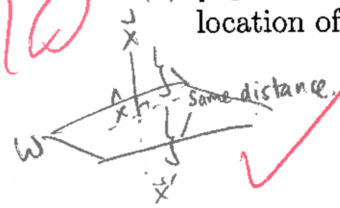
$$= \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-12}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$$

10+2

10

(d) [2-point BONUS]: Find the *mirror reflection* of $\mathbf{x}$ in the plane W, that is, the location of the image of $\mathbf{x}$ if W were a mirror [hint: draw a picture].



So $\vec{x} - \hat{x} = \hat{x} - \vec{x}' \Rightarrow \vec{x}' = 2\hat{x} - \vec{x}$

$$= 2\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 6 \\ -8 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ -1 \end{bmatrix}$$

8. [10 points] In each the following, 1 point is for true/false, and 1 point is for your explanation.

(a) True/false? A system of 6 equations in 7 unknowns that is consistent for *every* right-hand side vector can have *two* linearly-independent solutions.

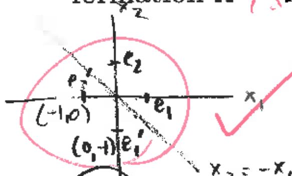
False

$6\begin{bmatrix} 6\times 7 \end{bmatrix}$ 7

when reduce A to echelon or reduced echelon form, A will have 6 pivots (because it's consistent for all $\vec{b}$ so pivots in every row)
There fore there's one free variable, so can have more than one solution.
But those solution might not be linearly independent

will

(b) True/false? A reflection through the line $x_2 = -x_1$ is given by the linear transformation $\mathbf{x} \to A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.



−1

True. We can construct A by stack up the transformatio
$\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$ which $= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in this case

(c) True/false? Given an $n \times n$ matrix A, if A^{-1} exists then the transformation $\mathbf{x} \to A\mathbf{x}$ is always an *isomorphism* of $\mathbb{R}^n$ to itself.

True. $\vec{x} \mapsto A\vec{x}$ is $\mathbb{R}^n$ to $\mathbb{R}^n$ that each $\vec{x}$ gives a unique $A\vec{x}$
in reverse $\vec{x} \mapsto A^{-1}\vec{x}$ is also $\mathbb{R}^n$ to $\mathbb{R}^n$ that each $\vec{x}$ gives a unique $A^{-1}\vec{x}$.
So $X \mapsto A\vec{x}$ is one-to-one and so onto $\mathbb{R}^n$.

(d) True/false? If the eigenvectors of a $n \times n$ matrix A form an orthogonal basis for $\mathbb{R}^n$, then A is always symmetric.

True. Since they are basis for $\mathbb{R}^n$, they have to be n L.I. eigenvectors.
We can normalize them to be orthonormal basis. Then stack them up to construct P
then $P^T = P^{-1}$. And $A = PDP^T$. And so A is symmetric. (A symmetric ⟺ orthogonally diagonalizable)
↳ D = matrix of eigenvalues.

good

(e) True/false? Given a linear dynamical system with matrix A, if all initial points $\mathbf{x}_0$ converge to the origin ($\mathbf{x}_k \to \mathbf{0}$ for $k \to \infty$), then all eigenvalues of A lie in the range $[0, 1)$.

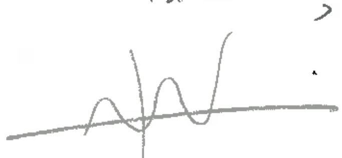
False the origin is the attractor when for all λ, $|\lambda| < 1$
so $\lambda \in (-1,1)$ not $[0,1)$

good

(f) [2-point BONUS:] True/false? Considered as a function of λ, det $(A - \lambda I)$ may be zero everywhere in some open (*i.e.* non-zero sized) interval of the real axis.

False, if this happens, A will have infinite eigenvalues
and A will have infinite dimensions.

yes



?? ← would force poly to be zero everywhere
but can't be since $(-\lambda)^n$
is present.

$\frac{10+1}{10}$