Math 22 Linear Algebra MIDTERM 1 summer, 2006

Abe B.

① a) aug. matrix $\left[\begin{array}{cccc|c} 2 & -6 & 1 & -2 & -1 \\ -1 & 3 & 2 & 6 & 3 \end{array}\right]$

$\sim \left[\begin{array}{ccccc} 1 & -3 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$ REF

x_1, x_3 basic variables (pivot columns)

x_2, x_4 free variables

crucial that you go all the way to write parametric form.

Write out REF as eqns:

$$\begin{aligned} x_1 &= +3x_2 + 2x_4 - 1 \\ x_2 &= x_2 \\ x_3 &= -2x_4 + 1 \\ x_4 &= x_4 \end{aligned}$$

"x_2 stuff" "x_4 stuff" "const stuff"

b) group by free variable $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ +1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

note it's free vars that become parameters

c) Homogeneous solution set has same free vars & coeff matrix in REF,

so $\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

You don't need to row reduce $[A \ \vec{0}]$ since you already did $[A \ \vec{b}]$ & $\vec{0}$ stays same

② a) True : linear systems have zero, one, or ∞ number of solutions. If there's 2 solutions then you're in the ∞ number category

b) False : to test for L.I., stack vectors in matrix, get $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = A$ a 2×3 matrix which can have at most 2 pivots

$\Rightarrow$ there's at least 1 free var $\Rightarrow$ $A\vec{x} = \vec{0}$ has non-trivial solutions

$\Rightarrow$ can't be Lin. Indep.

Linearity means $T(2\vec{u}) = 2T(\vec{u})$ for any $\vec{u}$, such as $\begin{bmatrix}1\\0\end{bmatrix}$.
But $T\left(2\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}6\\7\end{bmatrix} \neq \begin{bmatrix}6\\8\end{bmatrix} = 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ so T not linear

d) Let $C = AB$

Since C invertible, multiply from right by C^{-1}:

$$\underbrace{CC^{-1}}_{I} = (AB)C^{-1} = A(BC^{-1})$$ by associative property

so $AD = I$, for some D (namely BC^{-1})

$\rightarrow$ A is invertible by IMT (& it's inverse is BC^{-1})

* Note it's crucial you don't rely on A^{-1} existing before you've proved it!

Eg. $C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$ only holds if A^{-1} exists!

* Note also $B^{-1}AB \neq A$ since A, B need not commute!

* This was harder one, but in HW.

3) Stack 3 vectors into matrix $A = \begin{bmatrix}4 & -2 & 1\\ 2 & -1 & 0\\ -6 & 3 & h\end{bmatrix}$

row reduce $\sim \begin{bmatrix}4 & -2 & 1\\ 0 & 0 & -1/2\\ 0 & 0 & h-3/2\end{bmatrix}$ must continue to reduce — many forgot! $\sim \begin{bmatrix}4 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ EF

this happens for all values of h (including $3/2$).

a) $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{v}_3$ — "$\vec{b}$"

so interpret $\left[\begin{array}{cc|c}4 & -2 & 1\\ 2 & -1 & 0\\ -6 & 3 & h\end{array}\right] \sim \left[\begin{array}{cc|c}4 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{array}\right]$

as augmented matrix. The 2nd row $[0\ 0 \mid 1]$ means inconsistent, $\vec{v}_3$ cannot be written as lin. comb. of other 2, for $h \neq 0$ (or any h!).

b) Since EF has column 2 missing pivot, $A\vec{x} = \vec{0}$ not unique $\Rightarrow$ not Lin. Indep.

Dependence relation given by solution set of $\vec{x}$, best to get from R.E.F.

REF is $\begin{bmatrix}1 & -1/2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ so $\begin{aligned}x_1 &= \tfrac{1}{2}x_2\\ x_2 &= x_2 \text{ (free)}\\ x_3 &= 0\end{aligned}$ $\vec{x} = x_2\begin{bmatrix}1/2\\1\\0\end{bmatrix}$ ie $\tfrac{1}{2}\vec{v}_1 + \vec{v}_2 = \vec{0}$

c) To span $\mathbb{R}^3$ we need pivot in every row, which $\begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ does not have, for any h. $\Rightarrow$ does not span $\mathbb{R}^3$ for any h.

* Please use pivot rather than intuitive hand-waving arguments since they are rock solid and constitute proofs.

4 a) $ad-bc = 2(-3)-(-6) = -6+6 = 0$ so $\det A = 0$, not invertible.

b) $ad-bc = 7-6 = 1$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

c) Use $[A \mid I] \sim [I \mid A^{-1}]$

this is A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 18 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -38 & -32/3 & 17/3 \\ 0 & 1 & 0 & -16 & -13/3 & 7/3 \\ 0 & 0 & 1 & 7 & 2 & -1 \end{array}\right]$$

messy - sorry!

5 a) insert the three unit vectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$

defn: $A = [T(\vec{e_1})\ T(\vec{e_2})\ T(\vec{e_3})]$

$T(1,0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T(0,1,0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$T(0,0,1) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

stack as the columns.

$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

b) Check: onto means $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$ in $\mathbb{R}^2$, so need pivot in every row. A is already in E.F. with 2 pivots $\begin{bmatrix} \blacksquare & * & * \\ & \blacksquare & * \end{bmatrix}$

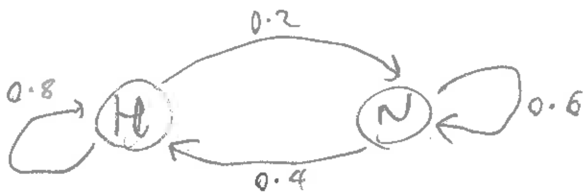
$\Rightarrow T$ is onto $\mathbb{R}^2$.

*Note: claiming T is projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ is not enough! Also, 'onto' is generally not related to Lin. Indep.!

c) One-to-one means $\vec{x}$ unique solution to $A\vec{x} = \vec{b}$, ie no free variables, ie pivot in every column. This doesn't hold, so T is not one-to-one.

* Note: you need to invoke pivots otherwise you have not explained it!

⑥ a)



check: diagonal element largest (since most of H stays in H; most of N stays in N)

$$M = \begin{array}{c} \\ H \\ N \end{array} \begin{array}{cc} H & N \\ \left[\begin{array}{cc} 0.8 & 0.4 \\ 0.2 & 0.6 \end{array}\right] \end{array}$$

col 1: various fractions H's pop goes each year

col 2: where N's pop goes each year

diagonal elements are the fractions that stay

b) $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in units of 10^4 (10,000) ← I recommend you use such rescaled units.

$$\vec{x}_1 = M\vec{x}_0 = \begin{bmatrix} 4/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 4/5 \end{bmatrix} \quad \text{ie} \quad \begin{array}{l} H_1 = 12{,}000 \\ N_1 = 8{,}000 \end{array}$$

c) $\vec{x}_2 = M\vec{x}_1 = M(M\vec{x}_0) = M^2\vec{x}_0$ so we want the matrix M^2 which takes $\vec{x}_0$ straight to $\vec{x}_2$, ie does 2 year's worth of 'evolution'.

$$M^2 = MM = \begin{bmatrix} 4/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix}\begin{bmatrix} 4/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix} = \frac{1}{5^2}\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 16+2 & 8+6 \\ 4+3 & 2+9 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} 0.72 & 0.56 \\ 0.28 & 0.44 \end{bmatrix}$$

do a sanity check: if this is valid migration matrix, each col. should total 1. They do.

* Note some of you did 3 years — I don't know why: please ask if in doubt.

* This question involved a little lateral thinking beyond the HW material (you never had to compute M^2 yet), but well within the course content so far. I like to push you a bit!