10/10/16

Your name: — SOLUTIONS —

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Math 22 Fall 2016, Midterm 1, Mon Oct 10

Please show your work. No credit is given for solutions without work or justification.

1. [8 points] Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 4 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

6 pts (a) Is the system consistent? If not, explain why. If consistent, write the general solution in *parametric vector form*:

Augmented matrix $\left[\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 4 & 0 & -2 \\ 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & -1 & -1 & 1 \end{array}\right] \leftarrow R_2 - 2R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ← EF: pivot in every row of coeff matrix, or zero rows, so, yes, consistent ← 3pts

Continue to REF: $\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ← x_3 free.

$$\begin{aligned} x_1 &= 1 + 2x_3 \\ x_2 &= -1 - x_3 \\ x_3 &= x_3 \end{aligned}$$

so $\vec{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}} t$, t any real number. ← 3pts

[Lose 1pt for arithmetic mistake]

2pts (b) Express the solution set to the corresponding *homogeneous* system $A\mathbf{x} = \mathbf{0}$ in the form of a span of one or more vectors:

For $A\vec{x} = \vec{0}$, solution set is as if $\vec{p} = \vec{0}$, ie $\vec{x} = \vec{v}t$ only.

Ie solution set = $\text{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$

Note: $x_3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is not a set, unless you write $\left\{x_3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, x_3 \text{ is real}\right\}$

2. [8 points] Consider the set of three vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 8 \\ 7 \\ h \end{bmatrix}$.

5pts (a) With $h = 3$, is the set linearly independent, and why? If not, give a dependence relation between $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Stack as columns of matrix $A = \begin{bmatrix} 1 & 2 & 8 \\ 2 & 1 & 7 \\ 3 & -1 & 3 \end{bmatrix}$

reduce:

$$\sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -3 & -9 \\ 0 & -7 & -21 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 8 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

is free variable, so $A\vec{x} = \vec{0}$ solution not unique.
$\Rightarrow$ not L.I. ← 3pts.

Continue to REF:

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution set to $A\vec{x} = \vec{0}$ is

$x_1 = -2x_3$
$x_2 = -3x_3$, ie $\vec{x} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} t$
$x_3 = x_3$

2pts

$$-2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}$$

gives coeffs in dependence relation

3pts (b) For what value(s) of h, if any, does the set span $\mathbb{R}^3$?

Reducing as before with general h gives: $\begin{bmatrix} 1 & 2 & 8 \\ 0 & -3 & -9 \\ 0 & -7 & h-24 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & \boxed{h-3} \end{bmatrix}$$

this entry must be a pivot so that A can have a pivot in every row, to span $\mathbb{R}^3$.

So: when $h-3 \neq 0$, ie $h \neq 3$.

3. [9 points]

(a) Give the definition of a square matrix A being invertible. [Note: do not give more than simply the definition.]

2pts

A is invertible if there exists a matrix D such that $AD = I$ and $DA = I$.

(b) Find the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, if it exists, or prove that it does not exist:

5pts

Let's proceed by trying to reduce $[A|I]$ to $[I|\text{something}]$, and if A does indeed reduce to I, by Thm 7 Ch.2.2, A is invertible, and its inverse is the "something" above:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

note it's easier not to scale last row yet.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

this is I_3, tells you A is invertible.

this is A^{-1}

2pts

(c) Write a linear system involving A whose solution gives the 2nd column of A^{-1}:

$$A\vec{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{e}_2}.$$

solution $\vec{x}$

Note: if you took $A\vec{x}$ for $\vec{x}$ = 2nd col of your A^{-1}, and didn't get $\vec{e}_2$, should worry!

4. [8 points]

(a) Give the definition of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ being one-to-one. [Note: do not give more than simply the definition.]

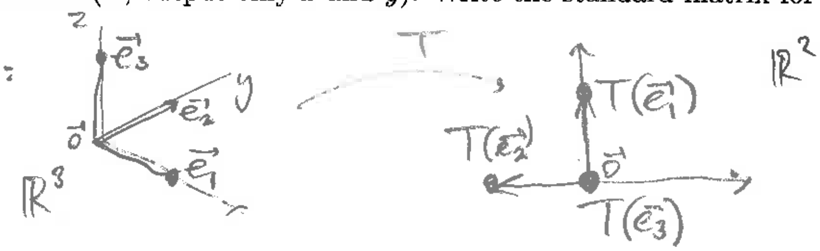
2pts.

T is one-to-one if, for each $\vec{b}$ in $\mathbb{R}^m$, there is at most one $\vec{x}$ in $\mathbb{R}^n$ whose image $T(\vec{x})$ is $\vec{b}$.

[typo, should be +.

2pts (b) Let a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ rotate points (x, y, z) by $+\pi/2$ about the z-axis (ie, 90° counterclockwise about the origin when viewed from above) then discard the z coordinate (ie, output only x and y). Write the standard matrix for T:

In pictures:



$$A = \left[T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)\right] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

check is $\underset{\text{output}}{2} \times \underset{\text{input}}{3}$

2pts (c) Is T from part (b) one-to-one? Why?

no, because std matrix A, when reduce to REF is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{-1} & 0 \end{bmatrix}$ which has x_3 as free variable.

Solution to $A\vec{x} = \vec{0}$ thus not unique $\Rightarrow$ are many $\vec{x}$ in $\mathbb{R}^3$ mapped to each $\vec{b}$ in the range.

2pts. (d) Is T from part (b) onto? Why?

Yes, since pivot in every row, $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^2$

5. [7 points]

(a) Let A and B be square matrices, and let AB be invertible. Prove, from only this information, that B is invertible. [You are allowed to use any of the theorems from class, but of course you must state when you use them.]

4pts.

By definition of invertibility, there is D such that $ABD = I$ and $DAB = I$. ← not useful.

Taking the 2nd eqn., we see $(DA)B = I$, so DA is a "left inverse" of B. By Invertible Matrix Theorem, if B has a matrix W such that $WB = I$, B is invertible. We choose $W = DA$. $\square$

Alternative using later material: $\det(AB) = (\det A)(\det B)$

$\det(AB)$ ≠ 0 since invertible; $\det B$ must not be zero, ⇒ B invertible.

(b) Let the 1st component h of a vector be the housed population, and the 2nd component s be the population living on the streets (homeless). Each year, 1% of those with houses become homeless. Also each year, half of the homeless find housing while the other half stays on the streets. Write the migration matrix which updates the population vector from year k to year $k+1$:

3pts.

$$\begin{bmatrix} h^{(k+1)} \\ s^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \begin{bmatrix} h^{(k)} \\ s^{(k)} \end{bmatrix}$$

−1pt if transposed.

BONUS: Say the total population is 5.1 million. Find the *equilibrium* population vector, ie one that does not change from year to year:

(1–2 pt)

call migration matrix A.

then want $\vec{x} = A\vec{x}$, ie $(A-I)\vec{x} = \vec{0}$, ← homog. sys; is nonunique?

ie $\begin{bmatrix} -0.01 & 0.5 \\ 0.01 & -0.5 \end{bmatrix} \begin{bmatrix} h \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ nonunique, solutions obey $-0.01h + 0.5s = 0$.

ie $h = 50s$

Ie, $h = \frac{50}{51}(5.1 \text{ million}) = \underline{5 \text{ million}}$, $s = \underline{0.1 \text{ million}}$.

6. [10 points] In this question only, no working is needed; just circle T or F.

(a) T / F: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if for each $\mathbf{x}$ in $\mathbb{R}^n$, there is a $\mathbf{b}$ in $\mathbb{R}^m$ such that $T(\mathbf{x}) = \mathbf{b}$.

It's backwards! Every T obeys the above! Read carefully.

(b) T / F: If A and B are invertible, then $((AB)^{-1})^T = (A^T)^{-1}(B^T)^{-1}$.

$$((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T = (A^T)^{-1}(B^T)^{-1}$$

(c) T / F: The solution set to a linear system can be changed by elementary row operations.

The whole point is that this set does not change.

(d) T / F: A linear transformation T is onto if every column of its standard matrix is a pivot column.

would need every row to have a pivot.

(e) T / F: Any set S of vectors containing the zero vector is linearly dependent.

the weight for it could be 1 & all other weights zero.

(f) T / F: For a matrix equation $A\boldsymbol{x} = \mathbf{0}$ to be consistent, the augmented matrix $[A \,|\, \mathbf{0}]$ must have a pivot in every row.

$A\vec{x} = \vec{0}$ is always consistent. Proof: Choose $\vec{x} = \vec{0}$.

(g) T / F: If any set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly dependent, then $\boldsymbol{v}_4$ can be written as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$.

all dependence relations could not involve $\vec{v}_4$

(h) T / F: A linear transformation T that maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ is a one-to-one map.

since std matrix is square $(n \times n)$, pivot in each row $\Leftrightarrow$ pivot in each col.

(i) T / F: If A is invertible, then $A\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution.

the opposite, it has only trivial solution.

(j) T / F: A map that takes the zero element to some other element cannot be a linear transformation.

It's the first test for linearity.