Math 22 Summer 2014 Midterm Exam II
Friday August 1, 2014

PRINT NAME: Solutions

INSTRUCTIONS: READ CAREFULLY!

This is a closed book, closed notes exam. Use of calculators is not permitted. You have two hours, do all problems.

On all **free response** questions below you must show your step-by-step work and make sure it is clear *how* you arrived at your solution. Whenever you answer a question, don't just say "Yes" or "No", but always justify your answers. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** Guessing is allowed: A wrong guess does not cost you more points than leaving the question unanswered. I will give partial credit on multiple choice, but not on T/F.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper.

FERPA waiver signature:

1. [10 pts] Let A be the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \end{pmatrix}$$

Find a basis for Col(A), Nul(A).

$$A \sim \begin{pmatrix} 1 & 3 & 2 & -6 \\ 0 & 0 & -5 & 23 \\ 0 & 0 & -5 & 21 \\ 0 & 0 & -10 & 44 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & -6 \\ 0 & 0 & -5 & 23 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \quad x = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{basis for Nul}$$

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 5 \\ 9 \\ 14 \end{pmatrix} \right\} \quad \text{basis for Col}$$

2. Given the matrices

$$A = \begin{pmatrix} 7 & 2 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 & 1 \\ 3 & 0 & 0 \\ 0 & -2 & -1 \end{pmatrix}, C = \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$$

(a) [5 pts] Use determinants to determine if A, B, C are invertible.

$\det(A) = 7 - 2 = 5$ ✓

$\det(B) = -3 \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = -3(-1+2) = -3$ ✓

$\det(C) = 12 - 12 = 0$

(b) [10 pts] Compute the inverse of all the invertible matrices from part (a).

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -1 & 7 \end{pmatrix}$$

$$B^{-1}: \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & -1/2 \\ -1 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & +1/2 & 1 & 1/3 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & -1 & -1/3 & -1 \\ 0 & 0 & 1 & 2 & 2/3 & 1 \end{array}\right)$$

B^{-1}

(c) [5 pts] Let S be the cube of side length 2 in $\mathbb{R}^3$, and S' its image under $T(x) = Bx$ where B is the 3×3 matrix as in part (a). What's the volume of S'?

(A) 3 (B) 24 (C) -24 (D) -3 (E) 8

$\mathrm{Vol}(S') = |\det(T)| \, \mathrm{Vol}(S) = 3 \cdot 8 = 24$

ANSWER

$\boxed{B}$

3. [10 pts]

The matrix

$$A = \begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

has eigenvalues $\lambda = -2, -1, 0$.

Diagonalize A by providing the matrices P, D. You do not need to compute P^{-1}

$$A + 0I = \begin{pmatrix} 2 & -2 & -2 & 0 \\ 3 & -3 & -2 & 0 \\ 2 & -2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A + I = \begin{pmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$$

$$A + 2I = \begin{pmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

4. Let H be the subspace spanned by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \\ 4 \end{pmatrix}, v_4 = \begin{pmatrix} 5 \\ 5 \\ 1 \\ 1 \end{pmatrix}, v_5 = \begin{pmatrix} 4 \\ 6 \\ -1 \\ 5 \end{pmatrix}$$

(a) [10 pts] Find a basis for H.

(b) [5 pts] What's the dimension of H? Is $H = \mathbb{R}^3$? Explain.

$$\begin{pmatrix} 1 & -2 & -1 & 5 & 4 \\ 2 & -1 & 1 & 5 & 6 \\ -2 & 0 & -2 & 1 & -1 \\ 3 & 1 & 4 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 & 5 & 4 \\ 0 & 3 & 3 & -5 & -2 \\ 0 & -4 & -4 & 11 & 7 \\ 0 & 7 & 7 & -14 & -7 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -2 & -1 & 5 & 4 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -2 & -1 & 5 & 4 \\ 0 & \boxed{1} & 1 & -2 & -1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

v_1, v_2, v_4 is a basis

H is a 3-dim subspace of $\mathbb{R}^4$, but it's <u>not</u> $\mathbb{R}^3$.

5. [10 pts] Diagonalize the matrix

$$A = \begin{pmatrix} -4 & 3 \\ 2 & 1 \end{pmatrix}$$

by providing the matrices P, D.

$$A - \lambda I = \begin{pmatrix} -4-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} \longrightarrow (-4-\lambda)(1-\lambda) - 6$$

$$= \lambda^2 + 3\lambda - 10$$

$$= (\lambda+5)(\lambda-2)$$

$$A + 5I = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$A - 2I = \begin{pmatrix} -6 & 3 \\ 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$\text{or} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -3 & 1 \\ 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}$$

6. (a) [10 pts] Show that

$$v_1 = \begin{pmatrix} 2 \\ -8 \\ 6 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 6 \\ -6 \end{pmatrix}$$

is a basis for $\mathbb{R}^3$. Justify your answer.

$$\begin{pmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -6 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Pivot in every column $\Rightarrow$ vectors are LI $\Rightarrow$ basis

(b) [5 pts] What are the coordinates of

$$x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

in the basis $B = \{v_1, v_2, v_3\}$ given in part (a)?

$(A)\ \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ $\quad (B)\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad (C)\ \begin{pmatrix} 2 \\ -1 \\ -7 \end{pmatrix}$ $\quad (D)\ \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ $\quad (E)\ \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

ANSWER

$\boxed{D}$

$$\begin{pmatrix} 2 & 3 & -1 & 0 \\ -8 & -7 & 16 & 1 \\ 6 & -1 & -6 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & -1 & 0 \\ 0 & 5 & 2 & 1 \\ 0 & -10 & -3 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 2 & 3 & -1 & 0 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 0 & 3 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

7. [5 pts] Given that

$$A = \begin{pmatrix} 3 & 17 & 9 & 4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 22 & 5 & 0 & 0 \\ 7 & 13 & 1 & 0 \\ 5 & 3 & 2 & -1 \end{pmatrix}$$

What is $\det(AB)$?

(A) 60 $\quad$ (B) 0 $\quad$ (C) 16 $\quad$ (D) -60 $\quad$ (E) -16

ANSWER

$\det(AB) = \det(A)\det(B) =$

$= -6 \cdot -10 = 60$

8. [5 pts] The matrix

$$A = \begin{pmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{pmatrix}$$

has $\lambda = 1$ as eigenvalue. Which of the following is an eigenvector corresponding to $\lambda = 1$?

$(A)\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad (B)\ \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ $\quad (C)\ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\quad (D)\ \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$ $\quad (E)\ \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

Either reduce A - I or realize that

$$A\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

ANSWER

$\boxed{E}$

TRUE or FALSE?

For each of the statements below indicate whether it is true or false. You do not need to justify your answers. For each incorrect or unanswered statement you'll lose 2 points. If five or more statements are incorrect you'll get 0 points out of the question.

9. [10 pt]

(a) **True / False** A diagonalizable matrix is invertible. False, invertible if and only if zero is not an eigenvalue

(b) **True / False** If A is a 4×6 matrix of rank 4, its null space has dimension 2. Rank-nullity theorem

(c) **True / False** If we swap two rows of a matrix, its determinant changes sign.

(d) **True / False** A linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^7$ has a 5×7 standard matrix.
It's a 7×5 matrix. You need to be able to multiply A by a vector in $\mathbb{R}^5$.

(e) **True / False** A 3×4 matrix can be invertible.
only square matrices are invertible

(f) **True / False** If A is a 2×2 matrix, $\det(2A) = 2\det(A)$. $\det(2A) = 2^2 \det(A)$

(g) **True / False** If an $n \times n$ matrix A is invertible, its column space is all of $\mathbb{R}^n$.

(h) **True / False** Eigenvectors corresponding to distinct eigenvalues are linearly independent.

(i) **True / False** Two matrices that have the same eigenvalues are similar. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ are not similar

(j) **True / False** A matrix with n eigenvalues is diagonalizable.
False: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has two eigenvalues but is not diag.