Your name:

Instructor (please circle): Alex Barnett Michael Musty

Math 22 Summer 2017, Final, Sunday Aug 27 / Monday Aug 28

Please show your work. No credit is given for solutions without work or justification.

1. [9 points] Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Which, if any, of the matrices A, B are invertible?

(b) Which, if any, of the matrices A, B have an eigenspace of dimension 2?

(c) Which, if any, of the matrices A, B are diagonalizable?

(d) Diagonalize every diagonalizable matrix from the previous part (i.e. find a diagonal D and invertible P so that the diagonalizable matrix equals PDP^{-1}. Do not compute P^{-1}).

2. [11 points] Consider the following web with three pages and links given by the diagram:

$$a \longrightarrow b \longleftarrow c$$

(a) Let A be the stochastic matrix for this web given by the PageRank algorithm (with the usual $\alpha = 1$). Find A, using the ordering a, b, c.

(b) Find the vector of importances for this web. Write this vector as a probability vector.

(c) Find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$. [Do not compute P^{-1}.]

(d) Must the Markov chain for A converge to your answer from (b), regardless of its initial probability vector ? Explain.

3. [7 points] Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

(a) Find an *orthogonal* basis for Col A.

(b) Find the coordinates of the point in Col A that is closest to the point $\begin{bmatrix} 0 \\ -2 \\ 5 \\ 4 \end{bmatrix}$.

4. [8 points] Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

(a) Solve the inconsistent system $A\mathbf{x} = \mathbf{b}$ in the least-squares sense.

(b) What is the smallest possible value of $\|A\mathbf{x} - \mathbf{b}\|$ for any $\mathbf{x} \in \mathbb{R}^2$?

(c) Write a matrix whose action on any vector in $\mathbb{R}^3$ is to orthogonally project it onto $(\mathrm{Col}\ A)^\perp$. [You may use a factored form to avoid writing all 9 entries.]

5. [7 points]

(a) Is $W = \left\{ \begin{bmatrix} s \\ t \\ t \end{bmatrix} : s, t \ \text{ real} \right\}$ a vector space? Prove your claim as succinctly as possible.

(b) Every element of the above W is in the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Is this set a basis for W ? Explain.

(c) Is the set $H := \{x \in \mathbb{R} : x \geq -1\}$ a vector space? Explain why.

6. [8 points] Short ones.

(a) For what real values of a is the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & a \\ 0 & 1 & a \end{bmatrix}$ invertible?

(b) An engineering problem produces A, a 2017×2019 matrix, and three linearly independent column vectors are known that when multiplied by A give the zero vector. Must there be a right-hand side $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution? Explain.

(c) If a 5×5 matrix has characteristic polynomial of the form $-\lambda^3(\lambda^2 + a\lambda + b)$, where $b \neq 0$, what are the possible values for its rank?

7. [9 points]

(a) Let A be an $n \times n$ matrix. Prove that the product of its eigenvalues equals its determinant. [Hint: write its characteristic polynomial two ways.]

(b) Let A be any $m \times n$ matrix. Prove that, if there exists a matrix B such that $BA = I$, where I is the $n \times n$ identity matrix, then A has linearly independent columns.

(c) Again let A be any $m \times n$ matrix. Prove that if A has linearly independent columns, then there exists a matrix B for which $BA = I$. [This is the converse of part (b), so together they prove: having L.I. columns is *equivalent* to possessing a "left inverse".]

BONUS: to what property is possessing a "right inverse" equivalent?

8. [10 points] Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$.

(a) What is the maximum $\|A\mathbf{x}\|$ can have over all of the unit vectors $\|\mathbf{x}\| = 1$?

(b) State a single geometric interpretation of the *smallest singular value* of A.

(c) Compute the full singular value decomposition of A (i.e. give U, Σ and V. Choose signs so that the top row of V has positive entries.)

9. [11 points] In this question only, no working is needed; just circle T or F.

(a) T / F: If A is row-equivalent to B, then $\operatorname{Col} A = \operatorname{Col} B$.

(b) T / F: If the linear system $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, then $\mathbf{b}$ can be written as a linear combination of the columns of A.

(c) T / F: If two vector spaces are isomorphic, they must have the same dimension.

(d) T / F: If a matrix A is diagonalizable, then every eigenvalue of A has algebraic multiplicity equal to 1.

(e) T / F: The set of continuous functions $f(t)$ on $0 \leq t \leq 1$ obeying $\int_0^1 f(t)dt = 0$ is a vector space.

(f) T / F: Let A be an $n \times n$ matrix. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. If we apply the Gram-Schmidt algorithm to $\mathcal{B}$, then the result is an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(g) T / F: The subset $\{1+2t^3, 2+t-3t^2, -t+2t^2-t^3\}$ is a linearly independent subset of $\mathbb{P}_3$.

(h) T / F: If A is a 5×3 matrix with rank 3, then it is impossible for $\operatorname{Nul} A$ to have strictly positive dimension.

(i) T / F: The zero vector is the only vector in $\mathbb{R}^n$ that satisfies $\|\mathbf{v}\| = \mathbf{v} \cdot \mathbf{v}$.

(j) T / F: Whenever A is an orthogonal matrix, so is A^3.

(k) T / F: In $\mathbb{R}^3$, the line $x = y = 0$ and the line $x = z = 0$ are orthogonal complements.