1. Determine whether each statement below is true or false and indicate your answer by circling the appropriate choice (1pt each):

(a) (True / False) If A is a 3×3 matrix with 2 pivot columns, then $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^3$.

(b) (True / False) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^n$, and $\mathbf{b}$ is also a vector in $\mathbb{R}^n$. Then $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b}$ has a solution.

(c) (True / False) The columns of a 2×3 matrix A must be linearly dependent.

(d) (True / False) The matrix $A = \begin{bmatrix} 4 & -h \\ h & 1 \end{bmatrix}$ is not invertible for some scalar h.

$\det A = 4 + h^2 > 0 \ \forall h$

(e) (True / False) If T is a linear transformation, then T either maps a line with parametric form $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, $t \in \mathbb{R}$, to a line or a point in its codomain.

2. Let $A = \begin{bmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(a) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Determine whether $A\mathbf{x} = \mathbf{b}$ is consistent or inconsistent. If $A\mathbf{x} = \mathbf{b}$ is consistent, describe its solution set in parametric vector form. If the system is inconsistent, explain why. (5pts)

$$[A\ \vec{b}] = \begin{bmatrix} 1 & 3 & 5 & 9 & 1 \\ 2 & 4 & 6 & 7 & 2 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} \underset{\substack{-2R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 3 & 5 & 9 & 1 \\ 0 & -2 & -4 & -11 & 0 \\ 0 & -1 & -2 & -5 & -1 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 3 & 5 & 9 & 1 \\ 0 & -1 & -2 & -5 & -1 \\ 0 & -2 & -4 & -11 & 0 \end{bmatrix}$$

$$\underset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} \boxed{1} & 3 & 5 & 9 & 1 \\ 0 & \boxed{-1} & -2 & -5 & -1 \\ 0 & 0 & 0 & \boxed{-1} & 2 \end{bmatrix}.$$

$A\vec{x} = \vec{b}$ is consistent b/c A has a pivot position in each row. Continuing to reduced echelon form:

$$\underset{\substack{-5R_3+R_2\to R_2 \\ 9R_3+R_1\to R_1}}{\sim} \begin{bmatrix} 1 & 3 & 5 & 0 & 19 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \underset{3R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & -14 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \underset{\substack{-R_2\to R_2 \\ -R_3\to R_3}}{\sim} \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & -14 \\ 0 & \boxed{1} & 2 & 0 & 11 \\ 0 & 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

$\Rightarrow$ Solns to $A\vec{x} = \vec{b}$ have the form

$$\vec{x} = \begin{bmatrix} -14 + x_3 \\ 11 - 2x_3 \\ x_3 \\ -2 \end{bmatrix},$$

where x_3 is free.

That is,

$$\boxed{\vec{x} = \begin{bmatrix} -14 \\ 11 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \forall t \in \mathbb{R}.}$$

(b) Is $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ in the span of the columns of A? Explain. (2pts)

Yes, b/c $A\vec{x} = \vec{b}$ is consistent.

(c) How many columns of A are pivot columns? (3pts)

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(d) Is $A\mathbf{x} = \mathbf{b}$ consistent for <u>every</u> $\mathbf{b}$ in $\mathbb{R}^3$? Explain. (3pts)

Yes, b/c A has a pivot position in each row.
(equivalently: last column of $[A\ \vec{b}]$ is not a pivot column for any $\vec{b} \in \mathbb{R}^3$).

3. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Determine the solution set in parametric vector form of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. (2pts)

x_2 is free, $x_3 = 0$, $x_1 = -2x_3 - x_2$
$= -x_2$.

So, $\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix}$, x_2 free. That is, $\boxed{\vec{x} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \ \forall t \in \mathbb{R}.}$

(b) The aim of the following statement is to provide a geometric description of the solution set found in part (a). Fill in the blanks to make the following statement true (2pts):

"The solution set of $A\mathbf{x} = \mathbf{0}$ obtained in part (a) is a line in $\mathbb{R}^3$ through the origin."

(c) Are the columns of A linearly independent? Explain. (2pts)

No, b/c 2 of the columns are equal & therefore one column is a linear comb. of the others.

(d) Let T be the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$, where A is the 3×3 matrix given at the beginning of the problem. Provide an explanation when answering the following 3 questions (2 pts each):

i. Is T linear?

Yes, b/c all matrix transformations are linear transformations.

ii. Is T onto $\mathbb{R}^3$?

No, b/c the columns of A (the standard matrix of T) do not span $\mathbb{R}^3$.

iii. Is T one-to-one?

No, b/c the columns of A are linearly dependent.

(e) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Is $A\mathbf{x} = \mathbf{b}$ consistent? Explain. (2pts)

No, b/c in this case the last column of $[A\ \vec{b}]$ is a pivot column.

(f) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Describe geometrically the solution set of $A\mathbf{x} = \mathbf{b}$ in relation to the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. (2pts)

$$[A\ \vec{b}] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\frac{1}{3}R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

. So, solns have the form $\vec{x} = \begin{bmatrix} 1/3 - x_2 \\ x_2 \\ 1/3 \end{bmatrix}$, where x_2 is free.

That is, $\vec{x} = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$.

The soln set of $A\vec{x} = \vec{b}$ is therefore a translation of the soln set in (a) by the vector $\vec{p} = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Compute the determinant of A. (3pts)

$$\det A = (1)\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - (0)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (1)(-1) - 0 + (1)(-1)$$

$$= -2$$

(b) Use elementary row operations to compute the inverse of A. (5pts)

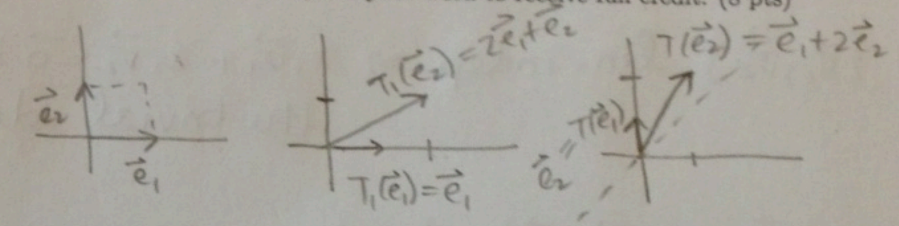
$$[A\ I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{-R_1+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \underset{-R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\underset{-\frac{1}{2}R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \underset{\substack{-R_3+R_2 \to R_2 \\ -R_3+R_1 \to R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$.

5. Determine the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first maps the unit vector $\mathbf{e}_1$ to itself and maps the unit vector $\mathbf{e}_2$ to the vector $2\mathbf{e}_1 + \mathbf{e}_2$, followed by a reflection across the line $x_2 = x_1$. Explain your work to receive full credit. (8 pts)

Graphically:



So, the standard matrix A of T is

$$A = [T(\vec{e}_1) \ T(\vec{e}_2)] = [\vec{e}_2 \ \ \vec{e}_1 + 2\vec{e}_1] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Algebraically: Reflecting $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $2\vec{e}_1 + \vec{e}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ across the line $x_2 = x_1$ means x_1 + x_2 entries are swapped so that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gets mapped to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ gets mapped to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Thus, $A = [T(\vec{e}_1) \ T(\vec{e}_2)] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

6. **BONUS PROBLEM** Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$. Prove that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is also linearly independent.

$\{\vec{v}_1, \vec{v}_2\}$ lin. indep. $\Rightarrow$ $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$ has only the trivial soln. (*)

$c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) = \vec{0}$ iff

$(c_1 + c_2)\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ iff

$c_1 + c_2 = 0$ and $c_2 = 0$ b/c of (*)

Now, $c_2 = 0 \Rightarrow c_1 = 0$.

Hence $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) = \vec{0}$ has only the trivial soln, which implies $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$ is linearly indep. $\square$