Math 22 Fall 2013 Midterm Exam I
Tuesday, October 9, 2013

PRINT NAME: Solutions

INSTRUCTIONS: READ CAREFULLY!

This is a closed book, closed notes exam. Use of calculators is not permitted. You have two hours, do all problems.

On all **free response** questions below you must show your step-by-step work and make sure it is clear *how* you arrived at your solution. Whenever you answer a question, don't just say "Yes" or "No", but always justify your answers. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** Guessing is allowed, and no points are subtracted for wrong answers.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FREE RESPONSE SECTION

Show your step-by-step work. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

1. (a) [5 pt] Solve the following system of equations:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & & & - & x_4 & = & 5 \\ 4x_1 & + & 2x_2 & + & 3x_3 & & & = & 11 \\ 2x_1 & + & x_2 & - & 15x_3 & - & 11x_4 & = & 0 \end{array}$$

(b) [3 pt] Describe the solution set in *parametric vector form.*

$$\begin{pmatrix} 2 & 1 & 0 & -1 & 5 \\ 4 & 2 & 3 & 0 & 11 \\ 2 & 1 & -15 & -11 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & -1 & 5 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & -15 & -10 & -5 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & -1 & 5 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1/2 & 0 & -1/2 & 5/2 \\ 0 & 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 = -\frac{1}{2}x_2 + \frac{1}{2}x_4 + \frac{5}{2} \\ x_3 = -\frac{2}{3}x_4 + \frac{1}{3} \\ x_2 \text{ free} \end{cases}$$

x_4 free

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/2 \\ 0 \\ -2/3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5/2 \\ 0 \\ 1/3 \\ 0 \end{pmatrix}$$

2. In items (a), (b), (c) below A is the 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 2 & 3 \\ 2 & 6 & 0 \end{pmatrix}$$

(a) [5 pt] Describe the solution set of the matrix equation $A\mathbf{x} = \mathbf{0}$ as a *Span* of vectors.

(b) [3 pt] Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for every possible vector $\mathbf{b}$ in $\mathbb{R}^3$?

(c) [3 pt] If the equation $A\mathbf{x} = \mathbf{b}$ is consistent (for a specific choice of $\mathbf{b}$), is the solution set a point, a line, a plane, or all of $\mathbb{R}^3$? Explain your answer.

(a) $$\begin{pmatrix} 0 & 1 & -3 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 6 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = -9x_3 \\ x_2 = +3x_3 \\ x_3 \text{ free} \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -9 \\ 3 \\ 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -9 \\ 3 \\ 1 \end{pmatrix} \right\}$$

(b) No, b/c the coefficient matrix A does not have a pivot in every row.

(c) Since the homogeneous solution is a line, the non-homogenous solution (when it exists) is also a line, translated by the particular solution associated to b.

3. Consider the following vectors in $\mathbb{R}^3$,

$$\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v_1} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(a) [5 pt] Is the vector $\mathbf{b}$ a linear combination of the three vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$?

(b) [3 pt] Do the three vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ span $\mathbb{R}^3$? Explain your answer.

(a) $$\begin{pmatrix} 3 & 2 & 0 & 1 \\ 5 & 3 & 1 & 4 \\ 4 & 3 & -1 & 0 \end{pmatrix} \overset{R_1 \to R_3 - R_1}{\sim} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 5 & 3 & 1 & 4 \\ 4 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -2 & 6 & 9 \\ 0 & -1 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The system is inconsistent b/c of the last row, so b is not a linear combination of $\{v_1, v_2, v_3\}$

(b) Since $b = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ is not a linear combination, $\{v_1, v_2, v_3\}$ do not span $\mathbb{R}^3$.

4. [5 pt] Determine if the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent. Explain your answer.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$Ax=0$ only has the trivial solution so a, b, c are LI.
(or, all columns are pivot, so a, b, c are LI).

5. (a) [5 pt] Write the plane $5x + 3y + z = 3$ in $\mathbb{R}^3$ in parametric vector form $\mathbf{x} = \mathbf{p} + t\mathbf{u} + s\mathbf{v}$.
Hint. Treat the equation as if it is a linear system with only one row.

(b) [3 pt] The set Span{$\mathbf{u}, \mathbf{v}$} is a plane in $\mathbb{R}^3$ as well. What is the *geometric* relation between the plane $5x + 3y + z = 3$ and Span{$\mathbf{u}, \mathbf{v}$}?

(a) $z = 3 - 5x - 3y \implies x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + x\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

so $p = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ $u = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

(b) $5x + 3y + z = 3$ is the plane Span{u,v} translated to go through the point $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ instead of the origin.

Note: While this is a reasonable midterm I question, it would be phrased as "Find the solution to $5x + 3y + z = 3$ and write it in parametric vector form $x = p + tu + sv$".

6. [5 pt] Consider the 2×2 matrix A,

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

Describe the set of all vectors $\mathbf{b}$ in $\mathbb{R}^2$ for which the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & b_1 \\ -4 & 2 & b_2 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{pmatrix}$$

The system is consistent only if $b_2 = 2b_1$, so the set of vectors is $b = \begin{pmatrix} b_1 \\ 2b_1 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

MULTIPLE CHOICE SECTION

Write your answer choice in the box that is provided. For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** *Guessing is allowed, and no points are subtracted for wrong answers.*

7. [5 pt] What is the standard matrix of the vertical shear depicted below?

Impossible to answer w/o the picture. Since it's a vertical shear it's either C or D.

$(A) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $(B) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $(C) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$(D) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $(E) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $(F) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

ANSWER

$\boxed{}$

8. [5 pt] What is the *reduced echelon form* of the matrix A below?

$$A = \begin{pmatrix} 0 & 2 & 0 & -2 \\ 1 & 2 & 3 & 0 \\ -1 & 0 & -3 & -1 \\ 2 & 7 & 6 & 0 \end{pmatrix}$$

$$(A) \begin{pmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (B) \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (C) \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(D) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (E) \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (F) \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ANSWER $\boxed{C}$

$$A \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & 0 & -1 \\ 0 & 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

10. [5 pt] Which of the following sets of vectors is linearly independent?

$$(A)\ \begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\quad (B)\ \begin{pmatrix}4\\-2\\-6\end{pmatrix},\begin{pmatrix}-6\\3\\9\end{pmatrix}\quad (C)\ \begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}-3\\0\end{pmatrix}$$

$$(D)\ \begin{pmatrix}2\\0\\4\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\3\\3\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\quad (E)\ \begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\3\\0\end{pmatrix},\begin{pmatrix}3\\3\\3\end{pmatrix}\quad \cancel{(F)}\ (F)\ \begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}-2\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$$

ANSWER

$\boxed{E}$

By exclusion:

$$\left.\begin{array}{l} (A)\ \text{has the } 0 \text{ vector} \\ (B)\ \begin{pmatrix}4\\-2\\-6\end{pmatrix} = -\tfrac{2}{3}\begin{pmatrix}-6\\3\\9\end{pmatrix} \\ (C)\ \begin{pmatrix}4\\0\end{pmatrix} = -\tfrac{4}{3}\begin{pmatrix}-3\\0\end{pmatrix} \\ (D)\ \text{Four vectors in } \mathbb{R}^3 \\ (F)\ (-2)\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\2\\0\end{pmatrix} \end{array}\right\}\ \text{LD Conditions}$$

Also, $\begin{pmatrix}1&2&3\\0&3&3\\0&0&3\end{pmatrix}$ is in echelon form already and has a pivot in every column

TRUE or **FALSE?**

For each of the statements below indicate whether it is true or false. You do not need to justify your answers. You get one point for each correct answer. Guessing will not hurt you: No points are subtracted for wrong answers.

11. [10 pt]

(a) **True / False** If the $m \times n$ matrix A has a pivot in every column then the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for every possible choice of $\mathbf{b}$ in $\mathbb{R}^m$. Need pivots in rows

(b) **True / False** If the matrix A has a pivot in every row then the columns of A are linearly independent.
Need pivot in columns

(c) **True / False** If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$ then A has a pivot in every row.

(d) **True / False** A system of 5 linear equations in 7 unknowns $x_1, x_2, \ldots, x_7$ never has a unique solution. At most 5 pivots for 7 columns $\Rightarrow$ free variable

(e) **True / False** A system of 5 linear equations in 7 unknowns $x_1, x_2, \ldots, x_7$ is always consistent.

(f) **True / False** The vector $\mathbf{v} - \mathbf{u}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

(g) **True / False** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$ then Span$\{\mathbf{u}, \mathbf{v}\}$ is always a plane in $\mathbb{R}^3$.
If $u = cv$ Span$\{u, v\}$ is a line

(h) **True / False** A set of 5 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ in $\mathbb{R}^4$ is always linearly dependent.
More vectors than entries

(i) **True / False** The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ is *onto* if the columns of the matrix A span $\mathbb{R}^m$.

(j) **True / False** The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ is *one-to-one* if the matrix equation $A\mathbf{x} = 0$ has nontrivial solutions.
It's one-to-one if A only has the trivial solution.