Solutions

Math 22, Exam I

April 22, 2010

NAME:

This is a closed book exam and you may not use a calculator. Use the space provided to answer the questions and if you need more space, please use the back of the exam making sure to write a note in the space provided that you have more work elsewhere that you would like me to grade. You must SHOW ALL WORK and be neat. If you have any questions, do not hesitate to ask.

Good luck!

Remember the honor code – do all of your own work.

1. Let

$$A = \begin{pmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

a. If consistent, solve the system $A\mathbf{x} = \mathbf{b}$ and write its solutions in parametric form. If it is not consistent, say so.

$$\begin{pmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{pmatrix}$$

System is inconsistent b/c of last row.

b. Solve the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 5 & 8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix} \sim \text{as above} \sim \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases} \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

c. Is the system $A\mathbf{x} = \mathbf{c}$ consistent for all $\mathbf{c} \in \mathbb{R}^3$? Explain.

No, it's inconsistent for $b = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ as in part (a).

2. Consider the three vectors

$$\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix}$$

a. Find h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{pmatrix}$$

If $h = -15$ the last row becomes $(0\ 0\ 3)$ and the system is inconsistent.

b. Find h such that the three columns of the above matrix are linearly independent.

No h will work, as there are more columns than rows, so we will always have a free variable in the homogeneous system.

Not a midterm I question.

3. Compute the determinants of the following matrices:

a.

$$\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$$

$\det = 10 - 12 = -2$

b.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

$\det = (-1)\Big((1)(3) - (4)(2)\Big) + (1)\Big((1)(2) - (3)(2)\Big) = -1$

c.

$$\begin{pmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & 0 & 2 \end{pmatrix}$$

$$\det = (2)\left(\det\begin{pmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right) = (2)(2)\left(\det\begin{pmatrix} 4 & 0 & 3 \\ 7 & 3 & 4 \\ 5 & 0 & 2 \end{pmatrix}\right) =$$

$$= (4)\Big((4)\big((3)(2) - (4)(0)\big) + (3)\big((7)(0) - (3)(5)\big)\Big) =$$

$$= 4 \cdot (24 - 45) = -84$$

Also not a midterm I question

4. Find the inverses of the following matrices.

a.

$$\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 6 & -10 \end{pmatrix}$$

b.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -5 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 16 & -7 & 4 \\ 0 & 1 & 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right) \qquad B^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 5 & -2 & 1 \\ -4 & 2 & -1 \end{pmatrix}$$

c.

$$\begin{pmatrix} 1 & 2 & -1 \\ 5 & -2 & 4 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 5 & -2 & 4 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -12 & 9 & -5 & 1 & 0 \\ 0 & 4 & 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & -12 & 9 & -5 & 1 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 15 & -2 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 13/15 & 1/15 & 3/15 \\ 0 & 1 & 0 & 19/60 & -1/30 & 3/20 \\ 0 & 0 & 1 & -2/15 & 1/15 & 3/15 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 14/60 & 2/15 & -1/10 \\ 0 & 1 & 0 & 19/60 & -1/30 & 3/20 \\ 0 & 0 & 1 & -2/15 & 1/15 & 3/15 \end{array}\right)$$

$$C^{-1} = \frac{1}{60} \begin{pmatrix} 14 & 8 & -6 \\ 19 & -2 & 9 \\ -8 & 4 & 12 \end{pmatrix}$$

Not an exam problem
(This is section 2.5 material)

5. Let

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$$

a. Find the LU decomposition of A.

$$A \sim \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} \sim \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = U$$

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix}$$

b. Let

$$B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Write B as a product of elementary matrices.

Since B is 2×2 the L matrix will be elementary, and up to a scalar the U matrix also will be

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

c Write B^{-1} as a product of elementary matrices.

$$B^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\right]^{-1} =$$

$$= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

a, b not midterm I questions

6. Answer the following questions by true or false:

a. The inverse of an elementary matrix is an elementary matrix.

True

b. The following matrix is invertible

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

False, $R_1 + R_2 = R_3$

c. Any linear system of equations whose coefficient matrix is of type 3×4 has a free variable.

True, we can have at most three pivot positions

d. I like linear algebra.

7. Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^2$ be the linear map given by

$$T(x_1, x_2, x_3) = (3x_2 - x_3, 2x_1 + x_2 + 3x_3).$$

a. What is the domain of T?

$\mathbb{R}^3$

b. What is the co-domain of T?

$\mathbb{R}^2$

c. What is the standard matrix for T?

$A = \begin{pmatrix} 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$, whose echelon form is $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$

d. Is T onto? Why or why not?

Yes, pivots in every row

e. Is T one-to-one? Why or why not?

No, there is a column w/o pivot.

8. Let $T : \mathbb{R}^2 \longmapsto \mathbb{R}^3$ be the linear transformation given by

$$T(\mathrm{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad T(\mathrm{e}_2) = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

a. Compute

$$T\begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

$$T\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3T(e_1) + 5T(e_2) = \begin{pmatrix} -2 \\ 6 \\ 29 \end{pmatrix}$$

b. Is $T : \mathbb{R}^3 \longmapsto \mathbb{R}^2$ given by $T(x_1, x_2, x_3) = (x_1 + 2x_3, x_1 + |x_2|)$ linear? Explain why or why not.

No. $T\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $-T\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ are different.

Not a midterm I question $\longrightarrow$

c. Suppose that A and B are $n \times n$ matrices such that both A and AB are invertible. Is B invertible?

Yes, in fact it's enough that AB is invertible to say that B is invertible.

Since AB is invertible, there is a matrix W such that $WAB = I$. But this means $(WA)B = I$, so there is a matrix such that $DB = I$. This is condition (j) of Theorem 8.