Barnett
7/19/17

SOLUTIONS

Your name:

Instructor (please circle): Alex Barnett Michael Musty

Math 22 Summer 2017, Midterm 1, Wed July 19

Please show your work. No credit is given for solutions without work or justification.

1. [8 points] Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

[6] (a) Is the system consistent? If not, explain why. If consistent, write the general solution in *parametric vector form*:

Augmented matrix $\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 2 \\ 2 & 4 & 1 & 1 & 2 \\ -1 & -2 & 0 & -1 & 0 \end{array}\right]$ swap rows 1 & 3 then negate row 1

$R_2 \leftarrow R_2 - 2R_1$ $\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$ $R_3 \leftarrow R_3 - R_2$ $\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ REF

x_2 free, x_4 free.

Read off rows: $x_1 = 0 - 2x_2 - 1x_4$, etc gives:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_4$$

x_2, x_4 any real numbers

could call s, t, etc

[2pts] (b) Express the solution set to the corresponding *homogeneous* system $A\mathbf{x} = \mathbf{0}$ in the form of a span of one or more vectors:

$$\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

notice the $\vec{p}$ "base point" is gone.

2. [9 points]

[4pts] (a) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 7 \\ -1 & 1 & -3 \end{bmatrix}$ if it exists, or prove that it does not exist:

Let's row reduce $[A \mid I]$

use R_3 as a "good" 2^{nd} row.

$R_2 \leftarrow R_3$
$R_3 \leftarrow R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 7 & 0 & 1 & 0 \\ -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right] \xrightarrow{\text{REF}} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -3 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right]$$

By Thm 7 in Ch. 2, since $A \sim I$ then A is invertible, and whatever I was turned into by the same row ops is A^{-1}.

$$A^{-1} = \begin{bmatrix} 10 & -3 & 3 \\ 1 & 0 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

[3pts] (b) Using just the definition of invertibility, prove that if A is invertible, the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for all right-hand sides (meaning, $\vec{b}$.). [Note: this is one of the parts of the Invertible Matrix Theorem, so you cannot use the IMT (or Thm 7, Ch. 2.) in your proof!]

By definition, A invertible means there is a C such that $CA = I$ & $AC = I$. Say A is $n \times n$.

Given any $\vec{b}$ in $\mathbb{R}^n$, we may right multiply both sides of $AC = I$ by $\vec{b}$ to get: $AC\vec{b} = I\vec{b} = \vec{b}$

Interpreting this equation, we see $\vec{x} = C\vec{b}$ solves $A\vec{x} = \vec{b}$ (check by substitution of $\vec{x}$: $A(C\vec{b}) = \vec{b}$ is the above equation). Since there is a solution $\vec{x}$ exhibited, it is consistent.

But. → Alternative & neat proof ii) by M. Kersey, which doesn't require power of IMT; just the defn:
Since A^2 inv, $CA^2 = I$. Left mult. by A & right-mult. by AC to get $ACAA^2C = A^2C$
Since $A^2C = I$ by defn. of inv. of A^2, then $ACA = I$, ie $B = CA$ obeys $AB = I$.
But also $BA = CAA = I$. Thus B satisfies the definition of A's inverse. ⇒ A is inv. ∎

[2pts] (c) Prove that if A^2 is invertible, and A is $n \times n$, then A is too.

Since A^2 is invertible, there is a C s.t. $CA^2 = I$

That is, $(CA)A = I$. This means $D = CA$ is a matrix satisfying $DA = I$. By the Invertible Matrix Theorem, A is invertible. [Note you need the power of the IMT this way, unlike in (b)]

Alternative proof i): $\det(A^2) = (\det A)^2$ so if one side is zero, the other is too, (note implicitly uses IMT since det relies on pivots).

3. [7 points]

[4pts] (a) Determine the value(s) of x so that the vectors $\begin{bmatrix} 1 \\ x \end{bmatrix}$ and $\begin{bmatrix} x \\ x+2 \end{bmatrix}$ are linearly independent:

Stack in matrix: $\begin{bmatrix} 1 & x \\ x & x+2 \end{bmatrix} \sim \begin{bmatrix} 1 & x \\ 0 & x+2-x^2 \end{bmatrix}$

pivot. ← is this a pivot? If so ⇔ set is L.I.

For the vectors to be L.I., $x+2-x^2 \neq 0$

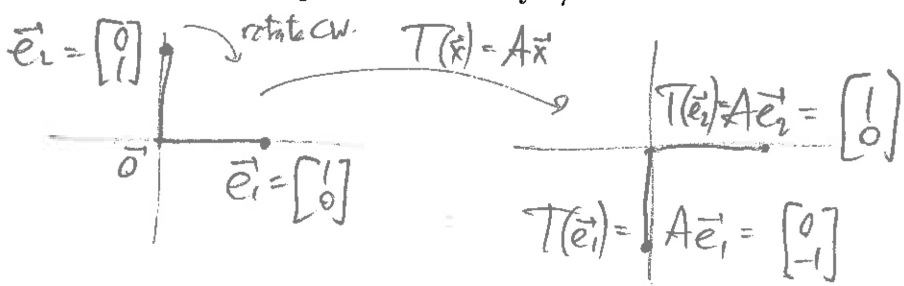
$\Rightarrow x^2 - x - 2 \neq 0$

$\Rightarrow (x-2)(x+1) \neq 0 \Rightarrow x \neq 2$ and $x \neq -1$

or: $x \in (-\infty,-1) \cup (-1,2) \cup (2,\infty)$

[3pts] (b) Find the standard matrix for the linear transformation from the plane to itself which rotates all points clockwise by $\pi/2$:

$\mathbb{R}^2$



"act on unit vectors then stack them"

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

4. [9 points] Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 3 & -5 & -10 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}.$$

[3] (a) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a set of vectors with each vector in $\mathbb{R}^n$. State the precise definition of what it means for this set of vectors to be linearly independent.

$\{\vec{v}_1, \cdots \vec{v}_4\}$ are linearly independent if the linear system $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$ has only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$.

[3] (b) Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ (consisting of the columns of A) linearly independent? If so, then prove your answer. If not, provide an explicit dependence relation.

Stack as columns then check no free variables.

$$\begin{bmatrix} 1 & -2 & -4 & 0 \\ 3 & -5 & -10 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

x_3 free

Not L.I.

$$\begin{aligned} x_1 &= 0 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \\ x_4 &= 0 \end{aligned} \quad \text{so} \quad \vec{x} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3$$

so: $0\vec{a}_1 - 2\vec{a}_2 + 1\vec{a}_3 + 0\vec{a}_4 = \vec{0}$ is a dependent relation.

[1] (c) Is T one-to-one? (yes or no will suffice)

No. (since its standard matrix has free variables)

[2] (d) Is T onto? Explain why or why not.

Yes, since its standard matrix has a pivot in every row, the range of T is all of $\mathbb{R}^3$, ie onto.

5. [7 points] Consider a human and zombie population. Each year a quarter of the humans become zombies and half the zombies die by various means. Let

$$\mathbf{x}_k = \begin{bmatrix} h_k \\ z_k \end{bmatrix}$$

be the state of the system k years after the zombie outbreak (where h_k is the human population k years after the zombie outbreak, and z_k is the zombie population k years after the zombie outbreak).

[4] (a) Find a 2×2 migration matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

no zombies turn back into humans.

$$\begin{bmatrix} h_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 3/4 & 0 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} h_k \\ z_k \end{bmatrix}$$

Note: humans do not die. (ironically it's the zombies who die).

col 1: what happens to humans.

col 2: what happens to zombies.

ie $A = \begin{bmatrix} 3/4 & 0 \\ 1/4 & 1/2 \end{bmatrix}$

[3] (b) Suppose $h_1 = 500$ and $z_1 = 275$. Use A^{-1} to find h_0 and z_0.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3/8 - 0}\begin{bmatrix} 1/2 & 0 \\ -1/4 & 3/4 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\text{so } \begin{bmatrix} h_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 4/3 & 0 \\ -2/3 & 2 \end{bmatrix}\begin{bmatrix} h_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 4/3 & 0 \\ -2/3 & 2 \end{bmatrix}\begin{bmatrix} 500 \\ 275 \end{bmatrix} = \begin{bmatrix} \frac{2000}{3} \\ 550 - \frac{1000}{3} \end{bmatrix} \approx \begin{bmatrix} 667 \\ 217 \end{bmatrix}$$

running dynamics backwards one year

BONUS Find the equilibrium vector $\mathbf{x}$ that is unchanged by multiplication by A.

Solve $\vec{x} = A\vec{x}$ ie $(A - I)\vec{x} = \vec{0}$

ie $\begin{bmatrix} -1/4 & 0 \\ 1/4 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ everyone died out :(

reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

the REF of transpose is not the transpose of the REF!

6. [10 points] In this question only, no working is needed; just circle T or F.

(a) T / F: The following is a valid proof that if A is invertible, so is A^T. The REF of A has a pivot in every row, so the REF of A^T would have a pivot in every column, so A^T row reduces to I, so A^T is invertible.

(b) T / F: The following is a valid proof that if A is invertible, its columns are linearly independent. Let $\mathbf{x}$ solve $A\mathbf{x} = \mathbf{0}$. Left-multiply by A^{-1} to get $\mathbf{x} = \mathbf{0}$. Thus the columns of A are linearly indepedent.

This is a needed part of the IMT, (a) $\Rightarrow$ (d).

(c) T / F: For any $n \times n$ matrix, the transpose of the inverse is the inverse of the transpose.

yes, $(A^T)^{-1} = (A^{-1})^T$

we assume both exist, otherwise question meaningless.

(d) T / F: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ being one-to-one means that T maps every $\mathbf{x}$ in $\mathbb{R}^n$ to a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

No, read carefully: this is only the defn of a map, ie a function. Would need: every $\vec{b}$ is image of at most one $\vec{x}$ in $\mathbb{R}^n$

(e) T / F: If the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ cannot be written as a linear combination of the columns of A.

since "$A\vec{x}$" means a linear combo of cols. of A.

(f) T / F: If the matrix equation $A\mathbf{x} = \mathbf{0}$ has a solution, then there is a dependence relation among the columns of A.

it always has the trivial soln $\vec{x} = \vec{0}$, even in the case of Lin. Dep. columns. Would need to be "solution other than $\vec{x} = \vec{0}$".

(g) T / F: $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 1 \\ x_1 + 1 \\ x_1 + x_2 \end{bmatrix}$ is linear.

$T(\vec{0}) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \neq \vec{0}$ vector of $\mathbb{R}^3$, so cannot be linear. needs $T(c\vec{u}) = cT(\vec{u})$

(h) T / F: It is possible to define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ that is one-to-one.

Yes,  5×3 can have pivot in every column.

(i) T / F: If the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, then $\mathbf{a}_1$ is always in Span$\{\mathbf{a}_2, \mathbf{a}_3\}$.

L.D. $\Rightarrow$ at least one vector is a lin. combo. of others, but cannot guarantee $\vec{a}_1$ is, so not always true. (as in book, only mark T if always true).

(j) T / F: If A is a square matrix, and the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is invertible.

Part of the IMT.

with grading rubric

Barnett
7/19/17

SOLUTIONS

Your name:

Instructor (please circle): Alex Barnett Michael Musty

Math 22 Summer 2017, Midterm 1, Wed July 19

Please show your work. No credit is given for solutions without work or justification.

1. [8 points] Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

[6] (a) Is the system consistent? If not, explain why. If consistent, write the general solution in *parametric vector form*:

since in this case we didn't ask to explain why, didn't deduct for using free vars to argue consistency. (!)

Augmented matrix $\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 2 \\ 2 & 4 & 1 & 1 & 2 \\ -1 & -2 & 0 & -1 & 0 \end{array}\right]$ swap rows 1&3 then negate row 1

$R_2 \leftarrow R_2 - 2R_1$ $\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$ $R_3 \leftarrow R_3 - R_2$ $\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ REF

x_2 free, x_4 free.

3 pts for this (incl. 1 for cons.)

Read off rows: $x_1 = 0 - 2x_2 - 1x_4$, etc gives:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_4$$

3 for interpreting & stacking

x_2, x_4 any real numbers

could call s, t, etc

since the correct numbers were crucial (& simple) we did take −1 for numerical mistake!

[2pts] (b) Express the solution set to the corresponding *homogeneous* system $A\mathbf{x} = \mathbf{0}$ in the form of a span of one or more vectors:

$$\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

notice the $\vec{p}$ "base point" is gone.

full 2pts if used vectors from (a) even if wrong; also if wrote the form correctly $\{s\vec{v}_1 + t\vec{v}_2, \ s,t \text{ real}\}$.

2. [9 points]

[4 pts] (a) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 7 \\ -1 & 1 & -3 \end{bmatrix}$, if it exists, or prove that it does not exist:

Let's row reduce $[A \mid I]$

use R_3 as a "good" 2^{nd} row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 7 & 0 & 1 & 0 \\ -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_3$
$R_3 \leftarrow R_2 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right] \xrightarrow{\text{REF}} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -3 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right]$$

By Thm 7 in Ch. 2, since $A \sim I$ then A is invertible, and whatever I was turned into by the same row ops is A^{-1}.

$$A^{-1} = \begin{bmatrix} 10 & -3 & 3 \\ 1 & 0 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

[3 pts] (b) Using just the definition of invertibility, prove that if A is invertible, the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for all right-hand sides. [Note: this is one of the parts of the Invertible Matrix Theorem, so you cannot use the IMT in your proof!]

meaning, $\vec{b}$.

By definition, A invertible means there is a C such that $CA = I$ & $AC = I$. Say A is $n \times n$.

Given any $\vec{b}$ in $\mathbb{R}^n$, we may right multiply both sides of $AC = I$ by $\vec{b}$ to get: $AC\vec{b} = I\vec{b} = \vec{b}$

Interpreting this equation, we see $\vec{x} = C\vec{b}$ solves $A\vec{x} = \vec{b}$ (check by substitution of $\vec{x}$: $A(C\vec{b}) = \vec{b}$ is the above equation).

Since there is a solution $\vec{x}$ exhibited, it is consistent.

1/3 if use IMT in your proof, or, equivalently Thm. 7. (this is (a) $\Leftrightarrow$ (b) in IMT).

[2pts] (c) Prove that if A^2 is invertible, then A is too.

Since A^2 is invertible, there is a C s.t. $CA^2 = I$

That is, $(CA)A = I$. This means $D = CA$ is a matrix satisfying $DA = I$. By the Invertible Matrix Theorem, A is invertible. [Note you need the power of the IMT here, unlike in (b)].

must state if use it!

Alternative proof: $\det(A^2) = (\det A)^2$ so if one side is zero, the other is too,

2/2 for this instead

3. [7 points]

[4pts] (a) Determine the value(s) of x so that the vectors $\begin{bmatrix} 1 \\ x \end{bmatrix}$ and $\begin{bmatrix} x \\ x+2 \end{bmatrix}$ are linearly independent:

+1 pt for determining a single value of x that works,

Stack in matrix: $\begin{bmatrix} 1 & x \\ x & x+2 \end{bmatrix} \sim \begin{bmatrix} 1 & x \\ 0 & x+2-x^2 \end{bmatrix}$ +2 for explanation

pivot. Is this a pivot? If so $\Leftrightarrow$ set is L.I.

For the vectors to be L.I., $x+2-x^2 \neq 0$

+2 for unsimplified quadratic expression

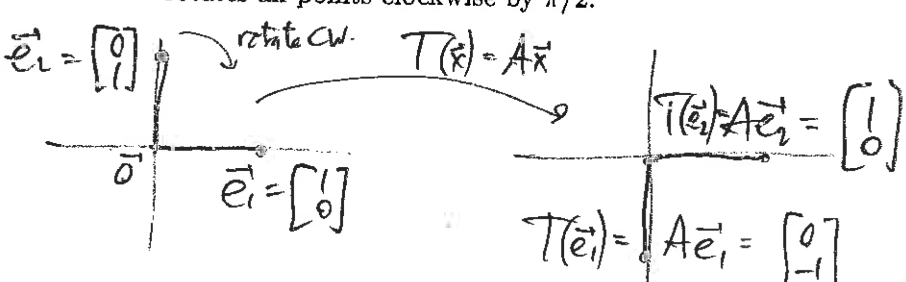
$\Rightarrow x^2 - x - 2 \neq 0$

or: $x \in (-\infty,-1) \cup (-1,2) \cup (2,\infty)$

$\Rightarrow (x-2)(x+1) \neq 0 \quad \Rightarrow x \neq 2$ and $x \neq -1$ +2 for correct values

[3pts] (b) Find the standard matrix for the linear transformation from the plane to itself which rotates all points clockwise by $\pi/2$:

$\mathbb{R}^2$



"act on unit vectors then stack them"

+1 for, +1 for, +1 realizing

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

+1 for attempting [T] or by rotation formula

+2 pts if computed anti-clockwise (correctly) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

4. [9 points] Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ with standard matrix

$$A = [\ \mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{a}_4\ ] = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 3 & -5 & -10 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

[3] (a) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a set of vectors with each vector in $\mathbb{R}^n$. State the precise definition of what it means for this set of vectors to be linearly independent.

$\{\vec{v}_1, \cdots \vec{v}_4\}$ are linearly independent if the linear system $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$ has only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$

+2 if wrote equivalent condition but not "the defn".

[3] (b) Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ (consisting of the columns of A) linearly independent? If so, then prove your answer. If not, provide an explicit dependence relation.

+2 if reasoned that columns are dependent but didn't include dependence relation.

Ok if just write dependence relation.

Stack as columns then check no free variables.

x_3 free

$$\begin{bmatrix} 1 & -2 & -4 & 0 \\ 3 & -5 & -10 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

+1pt for correct REF

Not L.I.

+1pt interpret REF

$x_1 = 0$
$x_2 = -2x_3$
$x_3 = x_3$
$x_4 = 0$.

so $\vec{x} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3$

so: $0\vec{a}_1 - 2\vec{a}_2 + 1\vec{a}_3 + 0\vec{a}_4 = \vec{0}$ is a dependent relation.

+1pt dependence relation

[1] (c) Is T one-to-one? (yes or no will suffice)

No. (since its standard matrix has free variables)

[2] (d) Is T onto? Explain why or why not.

+1pt explanation

Yes, since its standard matrix has a pivot in every row, the range of T is all of $\mathbb{R}^3$, ie onto.

+1pt

5. [7 points] Consider a human and zombie population. Each year a quarter of the humans become zombies and half the zombies die by various means. Let

$$\mathbf{x}_k = \begin{bmatrix} h_k \\ z_k \end{bmatrix}$$

be the state of the system k years after the zombie outbreak (where h_k is the human population k years after the zombie outbreak, and z_k is the zombie population k years after the zombie outbreak).

[4] (a) Find a 2×2 migration matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

no zombies turn back into humans

+2 for correct relationship not converted to "multipication by A".

$$\begin{bmatrix} h_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 3/4 & 0 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} h_k \\ z_k \end{bmatrix}$$

Note: humans do not die. (ironically it's the zombies who die).

+2 for transposing or permuting columns/rows

+2 pts for each column

col 1: what happens to humans.

col 2: what happens to zombies.

ie $A = \begin{bmatrix} 3/4 & 0 \\ 1/4 & 1/2 \end{bmatrix}$

[3] (b) Suppose $h_1 = 500$ and $z_1 = 275$. Use A^{-1} to find h_0 and z_0.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3/8 - 0}\begin{bmatrix} 1/2 & 0 \\ -1/4 & 3/4 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

+1 pt for computing inverse

+1 pt for algebra $A^{-1}A\vec{x}_0 = A^{-1}\vec{x}_1$

$$\text{so} \quad \begin{bmatrix} h_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 4/3 & 0 \\ -2/3 & 2 \end{bmatrix}\begin{bmatrix} h_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 4/3 & 0 \\ -2/3 & 2 \end{bmatrix}\begin{bmatrix} 500 \\ 275 \end{bmatrix} = \begin{bmatrix} \frac{2000}{3} \\ 550 - \frac{1000}{3} \end{bmatrix} \approx \begin{bmatrix} 667 \\ 217 \end{bmatrix}$$

running dynamics backwards one year.

+1 pt for correct computation of values with given A^{-1}.

BONUS Find the equilibrium vector $\mathbf{x}$ that is unchanged by multiplication by A.

+1 pt for bonus

Solve $\vec{x} = A\vec{x}$ ie $(A - I)\vec{x} = \vec{0}$

ie $\begin{bmatrix} -1/4 & 0 \\ 1/4 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ everyone died out :(

reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

the REF of transpose is not the transpose of the REF!

6. [10 points] In this question only, no working is needed; just circle T or F.

(a) T / F: The following is a valid proof that if A is invertible, so is A^T. The REF of A has a pivot in every row, so the REF of A^T would have a pivot in every column, so A^T row reduces to I, so A^T is invertible.

(b) T / F: The following is a valid proof that if A is invertible, its columns are linearly independent. Let $\mathbf{x}$ solve $A\mathbf{x} = \mathbf{0}$. Left-multiply by A^{-1} to get $\mathbf{x} = \mathbf{0}$. Thus the columns of A are linearly indepedent.

This is a needed part of the IMT, (a) $\Rightarrow$ (d).

(c) T / F: For any $n \times n$ matrix, the transpose of the inverse is the inverse of the transpose.

yes, $(A^T)^{-1} = (A^{-1})^T$

(d) T / F: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ being one-to-one means that T maps every $\mathbf{x}$ in $\mathbb{R}^n$ to a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

No, read carefully: this is only the defn of a map, ie a function.
Would need: every $\vec{b}$ is image of at most one $\vec{x}$ in $\mathbb{R}^n$

(e) T / F: If the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ cannot be written as a linear combination of the columns of A.

since "$A\vec{x}$" means a linear combo. of cols. of A.

(f) T / F: If the matrix equation $A\mathbf{x} = \mathbf{0}$ has a solution, then there is a dependence relation among the columns of A.

it always has the trivial soln $\vec{x} = \vec{0}$, even in the case of Lin. Dep. columns. Would need to be "solution other than $\vec{x} = \vec{0}$"

(g) T / F: $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 1 \\ x_1 + 1 \\ x_1 + x_2 \end{bmatrix}$ is linear.

$T(\vec{0}) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \neq \vec{0}$ vector of $\mathbb{R}^3$, so cannot be linear. needs $T(c\vec{u}) = cT(\vec{u})$

(h) T / F: It is possible to define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ that is one-to-one.

Yes, 5×3 can have pivot in every column.

(i) T / F: If the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, then $\mathbf{a}_1$ is in Span$\{\mathbf{a}_2, \mathbf{a}_3\}$.

L.D. $\Rightarrow$ at least one vector is a lin. combo. of others, but cannot guarantee $\vec{a}_1$ is.

(j) T / F: If A is a square matrix, and the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is invertible.

Part of the IMT.