The mathematics behind these games

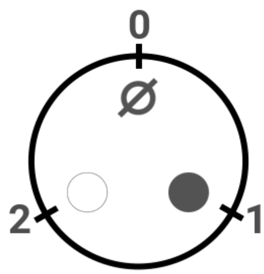
Abstract Algebra

Hidden behind the symbols empty field $\emptyset$, black piece $\bullet$ and white piece $\circ$ in these games are actual numbers. We have that

$$\boxed{\emptyset \text{ represents } 0\ ,\ \ \bullet \text{ represents } 1 \ \text{ and } \ \circ \text{ represents } 2.}$$

This way our table for addition can be written in the following way:

+	∅	●	○
∅	∅	●	○
●	●	○	∅
○	○	∅	●



+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Figure 1. Addition of numbers

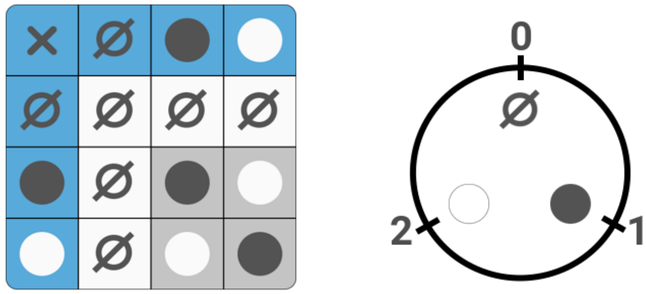
If we take a closer look we see that the addition is the usual addition except if the sum is larger or equal to 3. In fact we calculate as on a clock: If we pass $12 = 0$ on a clock we go back to 1. Seeing it in another way, if the result of our addition on a clock is larger than twelve, we subtract a multiple of twelve from the result such that the final result is a number between zero and twelve.

Figure 2. The way we add numbers on a clock is called addition modulo 12.
Example: 6 o'clock +14 hours = 8 o'clock.

Here we do the same, just with three numbers, meaning that we subtract a multiple of three if the result of our addition is larger than or equal to three. The same rule applies to multiplication, where we first multiply two numbers and then subtract a multiple of three to get either $0, 1$ or 2 as the final result.
In this case the only time we pass 3 is when we multiply $\circ \cdot \circ = 2 \cdot 2$. Again we subtract 3 and get

$$2 \cdot 2 = 4 = 3 + 1 = 1 \quad \text{or} \quad \circ \cdot \circ = \bullet.$$

Calculating this way is called *addition and multiplication modulo 3.* The set $\mathbb{F}_3$ of these three numbers $0, 1$ and 2 with the defined addition and multiplication $(\mathbb{F}_3, +, \cdot)$ is called the *prime field of order three.*



$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

FIGURE 3. Multiplication of numbers

Surprisingly this new number system follows all the rules that we are used to. For example:

1.) Addition and multiplication are commutative, meaning the order of how we add or multiply two numbers does not matter.
2.) There is a neutral element 0 for addition. If we add 0 to a number, we get the number back.
3.) There is a neutral element 1 in $\mathbb{F}_3$ without 0. If we multiply any non-zero number with 1 we get the number back.
4.) Every element has its inverse for addition. This means that for any number a there is another number b, such that $a + b = 0$. Can you pair the numbers in such groups?
5.) Multiplication is distributive over addition.

So our new number system $(\mathbb{F}_3, +, \cdot)$ is very similar to the numbers we are used to. You might ask yourself if there are more such number systems. The answer is yes! In fact for every prime number p we can make such a number system by calculating with the numbers $0, 1$ etc. to $p-1$ like on a clock with p ciphers. This way we obtain a number system $(\mathbb{F}_p, +, \cdot)$ that follows all the rules we are used to. However, to show this we would have to dive much deeper into the world of Abstract Algebra.

Linear Algebra

Solving linear equations

Linear Algebra is a branch of mathematics that deals - among other problems - with solving very simple equations. In such an equation only multiples and sums of variables can occur. Such an equation is called a *linear equation* and usually the task is to solve several such equations simultaneously. You might have encountered such problems at school. Maybe one like the following example.

Example The brothers Raman and Paman are together 23 years old. Raman is 3 years older than Paman. How old are Raman and Paman?
To solve this problem we can assign variables. We call x the age of Paman and y the age of Raman. Then we can turn our problem into two equations. From the description we obtain:

$$\begin{aligned} I.) \quad x + y &= 23 \\ II.) - x + y &= 3 \end{aligned}$$

Here we have two linear equations with the two variables x and y, which we want to solve simultaneously, meaning the solutions x and y must satisfy both equations. To find x and y we can solve one equation for a variable. For example, in this case we can use the second equation to obtain

$$II.) - x + y = 3 \text{ hence } x = y - 3.$$

Then we can replace x in the first equation to obtain

$$I.) \quad x + y = 23 \text{ and } x = y - 3 \text{ hence } (y-3) + y = 23 \text{ so } y = 13.$$

So Raman is 13 years old and as Paman is three years younger, he is 10 years old.

Another way of solving this is by noting that the following three operations are reversible and do not change the result of the solution:

1.) Multiplying both sides of an equation by a non-zero number.
Example: $I.) \quad x + y = 23$ is equal to $2 \cdot I.) \quad 2x + 2y = 46$.
2.) Adding an equation to another equation.
Example: $I.) + II.) \quad (x+y) + (-x+y) = 23 + 3$ or $2y = 26$. or $y = 13$.
3.) Swapping the order of two equations.

We can also combine 1.) and 2.) to obtain

4.) Adding the multiple of one equation to another equation.

We see in our example that 2.) gives the solution for y.

Question: Can we generalize this approach?

The answer is yes and the corresponding algorithm was already known more than 2000 years ago in ancient China. The method was rediscovered independently by Isaac Newton in 1670. It wrongly called *Gaussian elimination* after Carl Friedrich Gauss who used it to find the orbit of the dwarf planet Ceres. The method allows to solve several linear equations simultaneously. Coming back to our example we can write the information of the two equations in a a table where we assume that the x and y values in the different rows are added:

$$\begin{array}{l} I.) \quad x + y = 23 \\ II.) - x + y = 3 \end{array}$$

simplifies to

	x	y	sum
I.)	1	1	23
II.)	-1	1	3

or shorter $\left[\begin{array}{cc\|c} 1 & 1 & 23 \\ -1 & 1 & 3 \end{array}\right]$.

The idea is now to eliminate variables, meaning, we want an equation that contains only x or y and we want to achieve this goal by using the above four operations 1.) - 4.) which can actually replaced by just 3.) and 4.). We obtain a solution of our problem by applying the operations as shown below.

	x	y	sum
I.)	1	1	23
II.)	-1	1	3

$\overset{\text{Add}I.) \text{ to } II.)}{\Rightarrow}$

	x	y	sum
I.)	1	1	23
II.)	0	2	26

$\overset{1/2 \cdot II.)}{\Rightarrow}$

	x	y	sum
I.)	1	1	23
II.)	0	1	13

$\overset{\text{Add}-II.) \text{ to } I.)}{\Rightarrow}$

	x	y	sum
I.)	1	0	10
II.)	0	1	13

so $\begin{array}{l} x = 10 \\ y = 13 \end{array}$.

We can generalize this method to solve several equations with more variables simultaneously. In our first game we consider only the case where we have as many variables as equations. Also it is designed in a way that there is always a solution. In this case the solution can be obtained in the following way. We write down all equations in a table as above. Then we proceed in two phases, a forward and a backward phase.

Forward phase

1.) Find an equation or row that contains x i.e. the entry in the x column is different from zero. Swap this row such that it is positioned in the first row. Eliminate all other entries of x in the first column by adding a multiple of this first row to the other rows using 4.).
2.) Consider the the part of the table without the first column and row. Do the same for y what we did for x in 1.).
3.) Proceed as in 2.) by considering each time the table without the processed columns and rows. We obtain an upper triangular table.

We now multiply each row by the reciprocal of the leading non-zero number in the diagonal. This way all entries in the diagonal are ones. An example of a system of linear equations after the forward phase is shown below:

Example:

	x	y	w	z	sum
I.)	1	1	3	0	12
II.)	0	1	3	-2	4
III.)	0	0	1	-5	-3
IV.)	0	0	0	1	2

or shorter

$$\left[\begin{array}{cccc||c} \boxed{1} & 1 & 3 & 0 & 12 \\ 0 & \boxed{1} & 3 & -2 & 4 \\ 0 & 0 & \boxed{1} & -5 & -3 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{array}\right].$$

Backward phase

1.) Use the last variable in the last row to eliminate all its occurences in the rows above.
2.) Consider the the part of the table without the last column and row. Do the same for the second last variable as what we did for the last variable in 1.).
3.) Proceed as in 2.) by considering each time the table without the processed columns and rows. We obtain a table with only diagonal entries.

The solution can now be read from the table, where only the diagonal has non-zero entries. The solution to each variable is the column on the right. Our example matrix after the backward phase is shown below:

Example:

	x	y	w	z	sum
I.)	1	0	0	0	4
II.)	0	1	0	0	-13
III.)	0	0	1	0	7
IV.)	0	0	0	1	2

or

$$\left[\begin{array}{cccc||c} \boxed{1} & 0 & 0 & 0 & 4 \\ 0 & \boxed{1} & 0 & 0 & -13 \\ 0 & 0 & \boxed{1} & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{array}\right] \quad \text{so} \quad \begin{array}{c} x = 4 \\ y = -13 \\ w = 7 \\ z = 2 \end{array}.$$

This algorithm can also be applied in our first game. An example matrix from our game before and after the forward phase and backward phase is shown below.

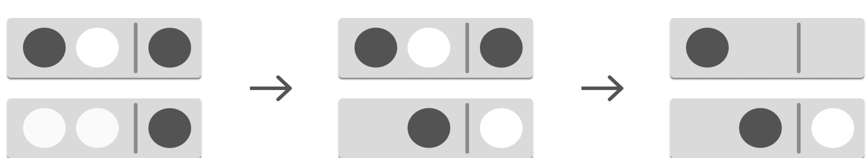


FIGURE 4. From left to right: A table from the game in starting position (left), after the forward phase and solved after the backward phase.

At the end the key we obtain is in fact the solution to a system of linear equations in our new number system $\mathbb{F}_3$.

Vectors, addition and dot product

Another important topic in Linear Algebra are vectors. A vector $\mathbf{v}$ of length n is an array with n entries which are numbers.

Example:

$$\mathbf{v} = (13, 2, 19) \text{ a vector of length three }, \quad \mathbf{u} = (1, 2, 4, 8) \text{ a vector of length four.}$$

We use vectors, when we want to look at arrays of data. The vector $\mathbf{v}$ in our example, could represent a date, for which we need three entries, day, month and year.
The solution of a linear equation can be seen as a vector. For example in our linear equation where we found the age of Paman and Raman, we could write $(x, y) = (10, 13)$ is the solution to our linear equation. Vectors of length two and three can be visualized in a two dimensional or three dimensional coordinate system.
The usual operations of vectors are entrywise. For example, we can add vectors of the same size by adding the corresponding entries.
Example:

$$\mathbf{v} = (4, -2, 3), \mathbf{u} = (1, 0, -2) \text{ then } \mathbf{v} + \mathbf{u} = (4 + 1, -2 + 0, 3 + (-2) = (5, -2, 1).$$

We can also multiply a vector with a number. Again, that means we have to multiply every entry with this number:
Example:

$$\mathbf{v} = (4, -2, 3), \text{ then } 3 \cdot \mathbf{v} = (3 \cdot 4, 3 \cdot -2, 3 \cdot 3) = (12, -6, 9).$$

The same rules apply to our vectors in the second game. To play this game we have to find the vectors that add up to so called zero vectors, where all entries are zero. If a falling vector falls onto another, such that these two add up to the zero vector, then the two vectors disappear. This is shown below:
If a vector lands over an empty field in our game, then it can not vanish - the falling vector

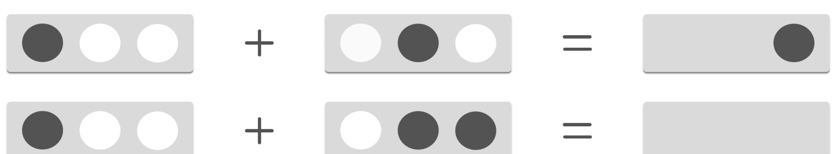


FIGURE 5. Vector addtion of row vectors. In the second example the two vectors add up to the zero vector.

and the vector it falls onto do not have the right size.

Besides these two operations for vectors there is another operation, called the *dot product*. This is a bit more complicated. We can find the dot product of two vectors of the same size in two steps. First we "multiply" two vectors entrywise and then we add these entries together. To better depict this operation, we write the vectors in columns:
Example:

$$\mathbf{v} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \text{ then } \mathbf{v} \bullet \mathbf{u} = \begin{pmatrix} 4 \cdot 1 \\ -2 \cdot 0 \\ 3 \cdot -2 \end{pmatrix} \begin{matrix} + \\ + \end{matrix} = 4 \cdot 1 + (-2) \cdot 0 + 3 \cdot -2 = -2.$$

The same rules apply to our vectors in the third game. To play this game we have to find the vectors whose dot product is zero. If a falling vector falls onto another, such that the dot product of these two vectors is zero, then the two vectors disappear. An example is shown below. Again, if a vector lands over an empty field in our game, then it can not vanish.

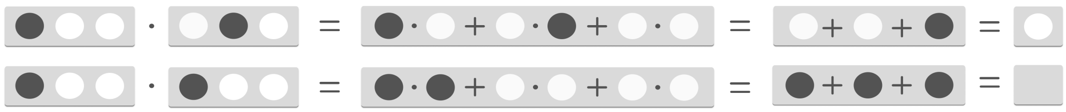


FIGURE 6. Dot product between row vectors. In the second example the dot product is $\emptyset = 0$.