# Math 22: Linear Algebra
# Fall 2019 - Homework 4

**Total:** 20 points Return date: Wednesday 10/16/19

Numbered problems are taken from Lay, D. et al: *Linear Algebra with Applications*, fifth edition.
Please show your work; no credit is given for solutions without work or justification.

## Part A

1. Let $A$ be an $n \times n$ matrix. Complete the proof of **Lecture 10, Theorem 5***. Show that:
for each $\mathbf{b} \in \mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\Rightarrow A$ is invertible.
You should argue using the following steps:

   a) Show that $AC = I_n$ for some matrix $C$. **Hint:** Use the unit vectors.

   **Solution:** We know that $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$. Therefore this equation has also solutions for the unit vectors. We collect these solutions:

   $$A\mathbf{c}_1 = \mathbf{e}_1\ ,\ A\mathbf{c}_2 = \mathbf{e}_2, \ldots, A\mathbf{c}_n = \mathbf{e}_n.$$

   Then we set $C = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$, the matrix whose columns are the $\mathbf{c}_i$. The above equations imply for the matrix

   $$[A\mathbf{c}_1, A\mathbf{c}_2, \ldots, A\mathbf{c}_n] = AC = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = I_n. \quad \text{So} \quad AC = I_n.$$

   b) Show that for the matrix $C$ from part a) we have: $C\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$.

   **Solution:** We have $C\mathbf{x} = \mathbf{0} \Rightarrow \underbrace{AC}_{=I_n}\mathbf{x} = C\mathbf{0} = \mathbf{0} \Rightarrow \mathbf{x} = I_n\mathbf{x} = \mathbf{0}$.

   c) Explain why for each $\mathbf{b} \in \mathbb{R}^n$ the equation $C\mathbf{x} = \mathbf{b}$ has a unique solution.

   **Solution:** Part b) implies that the map $C\mathbf{x}$ is one-to-one. That means that $C$ has a pivot in every column. As $C$ is a square matrix, it has a pivot in every row. So the map $C\mathbf{x}$ is both one-to-one and onto. This implies our statement.

   d) Show that there is a matrix $B$, such that $CB = I_n$, then show that $B = A$.

   **Solution:** By part c) we have like for $A$ in part a) that there is a matrix $B$, such that $CB = I_n$. So $AC = I_n$ and $CB = I_n$. Therefore $\underbrace{AC}_{=I_n}B = AI_n$, but that means that $A = B$. So $CA = I_n$.

   Part a) and d) together then imply that $A$ is invertible and that $C = A^{-1}$.
   **Note:** You should not just cite the **Invertible Matrix Theorem** but instead prove these statements on your own.

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2. §2.3.12 True / False questions. All matrices are $n \times n$ matrices.

   a) If there is an $n \times n$ matrix such that $AD = I_n$, then there is an $n \times n$ matrix such that $CA = I_n$.
   **True** This follows from the **Invertible Matrix Theorem**.

   b) If the columns of $A$ are linearly independent, then the columns of $A$ span $\mathbb{R}^n$.
   **True** Again, this follows from the **Invertible Matrix Theorem**.

   c) If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each vector $\mathbf{b}$ in $\mathbb{R}^n$, then the solution is unique for each $\mathbf{b}$ in $\mathbb{R}^n$.
   **True** This is another consequence of the **Invertible Matrix Theorem**.

   d) If the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then $A$ has $n$ pivot positions.
   **False** $A$ could be any square $n \times n$ matrix and not all of them have $n$ pivot positions.

   e) If there is a vector $\mathbf{b}$ in $\mathbb{R}^n$, such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.
   **True** If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent then $A$ does not have a pivot in every row. As $A$ is a square matrix it does not have a pivot in every column. That implies that $T$ is not one-to-one.

3. §2.3.18 Let $C$ be a $6 \times 6$ matrix such that the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every vector $\mathbf{v}$ in $\mathbb{R}^6$. Can there be a vector $\mathbf{v}$, such that the equation $C\mathbf{x} = \mathbf{v}$ has more than one solution?

   **Solution:** This is not possible. The condition that $C\mathbf{x} = \mathbf{v}$ has always a solution implies that the map $\mathbf{x} \mapsto C\mathbf{x}$ is onto. As $C$ is a square matrix this means that the map is also one-to-one. This means that there is always a unique solution $\mathbf{x}$ for the equation $C\mathbf{x} = \mathbf{v}$.

   **Part B**

4. §2.3.32 Let $A$ be an $n \times n$ matrix such that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Without using the **Invertible Matrix Theorem**, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$.

   **Solution:** As the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution we know that $A$ has a pivot in every row. As $A$ is a square matrix, it also has a pivot in every column. But that means that the map $\mathbf{x} \mapsto A\mathbf{x}$ is onto. Hence the equation $A\mathbf{x} = \mathbf{b}$ must have a solution $\mathbf{x}$ for each $\mathbf{b}$.

5. §3.1.8 Compute the determinant using a cofactor expansion across the first row.

   **Solution:** Expanding along the first row we get:

$$\begin{vmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix} = 4 \cdot \begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 4 & 3 \\ 6 & 5 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 0 \\ 6 & 1 \end{vmatrix} = 4 \cdot 6 + (-1) \cdot 11 + 2 \cdot (-8) = -3.$$

6. §3.1.10 Compute the determinant using a cofactor expansion. At each step use the most efficient expansion.

   **Solution:** We start by expanding across the second row, as this row has the most zeros.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix} \overset{\text{Row 2}}{=} (-3) \cdot \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix} \overset{\text{Row 3}}{=} (-3) \cdot \left( 2 \cdot \begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix} + 5 \cdot \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} \right)$$
$$= (-3) \cdot (2 \cdot (-2) + 5 \cdot 0) = 12.$$

**Part C**

7. §3.2.8 Find the determinant by row reduction to echelon form.

   **Solution:** We row reduce the matrix to echelon form $U$. In each step we record the operation performed and how it scales the determinant. We then multiply this number with the determinant of $U$.

$$A = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix} \overset{-2R1+R3, 3R1+R4}{\rightarrow} \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix}$$
$$\overset{-R2+R3, R2+R4}{\rightarrow} \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -10 \end{bmatrix} \overset{R3 \leftrightarrow R4}{\rightarrow} \begin{bmatrix} \boxed{1} & 3 & 2 & -4 \\ 0 & \boxed{1} & 2 & -5 \\ 0 & 0 & \boxed{1} & -10 \\ 0 & 0 & 0 & \boxed{10} \end{bmatrix} = U.$$

   As $U$ is a diagonal matrix its determinant $\det(U) = 1 \cdot 1 \cdot 1 \cdot 10 = 10$. Adding a row to another row does not change the determinant. Only the last step in the row reduction, where we swap a row changes the sign of the determinant. Therefore

$$\det(A) = -\det(U) = -10.$$

8. §3.2.18 Let $A$ be the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ where } \det(A) = 7. \text{ Find } \det(B), \text{ where } B = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

   **Solution:** We obtain $A$ from $B$ by swapping row 1 and row 2. Therefore these two matrices differ only by a factor of $-1$. We have $\det(B) = -\det(A) = -7$.

**Math 22: Linear Algebra**
**Fall 2019 - Homework 4**

**Total:** 20 points — Return date: Wednesday 10/16/19

9. §3.2.28 True / False questions.

   a) If three row interchanges are made in succession, then the new determinant equals the old determinant.
   **False** If we do three row interchanges in the matrix $A$ to obtain the matrix $B$, then $\det(B) = (-1)^3 \det(A) = -\det(A)$.

   b) The determinant of $A$ is the product of the diagonal entries in $A$.
   **False** This is not true in general. **Counterexample:**
   $$\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 0 \neq 2 \cdot 3 = 6.$$

   c) If $\det(A) = 0$ then two rows or two columns are the same, or a row or a column is zero.
   **False** This is not true in general. **Counterexample:**
   $$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0.$$

   d) $\det(A)^{-1} = (-1)\det(A)$.
   **False** $\det(A)^{-1} = \dfrac{1}{\det(A)}$.

**Part D**

10. §4.1.2 Let $W$ be the union of the first and third quadrants in the $xy$-plane. That is let
$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \text{ such that } x \cdot y \geq 0 \right\}.$$

   a) If $\mathbf{u}$ is in $W$ and $c$ is a number, is $c \cdot \mathbf{u}$ in $W$? Why?
   **Solution:** This is true.
   $$\text{If } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ then } c \cdot \mathbf{u} = \begin{bmatrix} c \cdot u_1 \\ c \cdot u_2 \end{bmatrix} \text{ and } (c \cdot u_1)(c \cdot u_2) = c^2(u_1 \cdot u_2).$$
   But as by definition of $W$ we have $u_1 \cdot u_2 \geq 0$, we also have $(c \cdot u_1)(c \cdot u_2) = c^2(u_1 \cdot u_2) \geq 0$. That means that $c \cdot \mathbf{u}$ is also in $W$.

   b) Find vectors $\mathbf{u}$ and $\mathbf{v}$, such that $\mathbf{u} + \mathbf{v}$ is not in $W$.
   **Solution:** For $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we have that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is not in $W$. This means that $W$ can not be a vector space as it does not satisfy the subspace criteria.

11. §4.1.6 Determine whether all polynomials of the form $p(t) = a + t^2$, where $a$ is in $\mathbb{R}$ is a subspace $H$ of $\mathbb{P}_2$, the space of polynomials of degree two.
**Solution:** $H$ is not a subspace, as $q(t) = t^2$ in $H$, but
$$2 \cdot t^2 \text{ is not in } V.$$

**Math 22: Linear Algebra**
**Fall 2019 - Homework 4**

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12. §4.1.8 Determine whether all polynomials in $\mathbb{P}_n$, where $p(0) = 0$ form a subspace $H$ of $\mathbb{P}_n$.
**Solution:** $H$ is a subspace. We check the subspace criteria. Let $p, q$ be in $H$ and $c$ in $\mathbb{R}$. Then we have to show that

    i) The zero polynomial $O : t \to O(t) = 0$ is in $H$,as $O(0) = 0$.
    ii) $p + q$ in $H$: We check: $(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$, so $p + q$ in $H$.
    iii) $c \cdot p$ in $H$: We check: $(c \cdot p)(0) = c \cdot p(0) = 0$, so $c \cdot p$ in $H$.

13. §4.1.32 Let $H$ and $K$ be subspaces of a vector space $V$. Show that the intersection $H \cap K$ is a subspace of $V$. Then give a counterexample in $\mathbb{R}^2$ that the union $H \cup K$ is not, in general, a subspace.
**Solution:** We first look at $H \cap K$. To show that $H \cap K$ is a subspace we check the subspace criteria. Let $\mathbf{v}, \mathbf{w}$ be in $H \cap K$ and $c$ in $\mathbb{R}$. We know that

$$\mathbf{v} \in H \cap K \Leftrightarrow \mathbf{v} \in H \text{ and } \mathbf{v} \in K.$$

Then we have to show that

    i) The zero vector $\mathbf{0}$ is in $H \cap K$: As $\mathbf{0} \in H$ and $\mathbf{0} \in K$ - as both are subspaces - we know that $\mathbf{0} \in H \cap K$.
    ii) $\mathbf{v} + \mathbf{w}$ in $H \cap K$: We know that

$$\mathbf{v} \in H \cap K \Rightarrow \mathbf{v} \in H \text{ and } \mathbf{w} \in H \cap K \Rightarrow \mathbf{w} \in H.$$

As $\mathbf{v} \in H$ and $\mathbf{w} \in H$ we know - as $H$ is a subspace that $\boxed{\mathbf{v} + \mathbf{w} \in H}$. Similarly

$$\mathbf{v} \in H \cap K \Rightarrow \mathbf{v} \in K \text{ and } \mathbf{w} \in H \cap K \Rightarrow \mathbf{w} \in K.$$

Hence $\boxed{\mathbf{v} + \mathbf{w} \in K}$. But that implies that $\mathbf{v}+\mathbf{w}$ in both subspaces, therefore $\mathbf{v}+\mathbf{w} \in H \cap K$.

    iii) $c \cdot \mathbf{v}$ in $H \cap K$: We know that

$$\mathbf{v} \in H \cap K \Rightarrow \mathbf{v} \in H \Rightarrow c\mathbf{v} \in H,$$

where the last implication follows from the facth that $H$ is a subspace. By the same reasoning $c\mathbf{v} \in K$. But that implies that $c\mathbf{v}$ in both subspaces, therefore $c\mathbf{v} \in H \cap K$.

As a counterexample take the lines $L_1 = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ and $L_2 = \text{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$. It is easy to check that these are both subspaces of $\mathbb{R}^2$. Consider $L_1 \cup L_2$. We have that

$$\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \in L_1 \cup L_2 \text{ but } \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \text{ is not in } L_1 \cup L_2.$$

Therefore $L_1 \cup L_2$ is **not** a subspace.