Math 22: Linear Algebra
Fall 2019 - Homework 5

Total: 20 points Return date: Wednesday 10/23/19

Numbered problems are taken from Lay, D. et al: *Linear Algebra with Applications*, fifth edition.
Please show your work; no credit is given for solutions without work or justification. For questions marked with a **(C)** a computer algebra system can be used and intermediate steps are not asked.

Part A

1. §4.2.6 Find an explicit description of $\mathrm{Nul}(A)$, where

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: $\mathrm{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^5, \text{ such that } A\mathbf{x} = \mathbf{0}\}$. So we have to solve the system of linear equations with augmented matrix $[A|\mathbf{0}]$. We get:

$$[A|\mathbf{0}] = \left[\begin{array}{ccccc|c} 1 & 5 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{-R2+R1}{\to} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -6 & -8 & 1 & 0 \\ 0 & \boxed{1} & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{ref}([A|\mathbf{0}]).$$

From the reduced echelon form we read out the solution. We obtain:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \cdot \underbrace{\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{=\mathbf{v}_1} + x_4 \cdot \underbrace{\begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{=\mathbf{v}_2} + x_5 \cdot \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{=\mathbf{v}_3}, x_3, x_4, x_5 \in \mathbb{R}.$$

So $\mathrm{Nul}(A) = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

2. §4.2.10 Determine whether the set W given below is a vector space or not.

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4, \text{ such that } c = a + 3b, d = a + b + c \right\}.$$

Solution: We can rewrite c and d in terms of the other variables and see:

$$W = \left\{ \begin{bmatrix} a \\ b \\ a+3b \\ a+b+(a+3b) \end{bmatrix}, a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} a \\ b \\ a+3b \\ 2a+4b \end{bmatrix}, a, b \in \mathbb{R} \right\} = \left\{ a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}}_{=\mathbf{u}_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}}_{=\mathbf{u}_2}, a, b \in \mathbb{R} \right\}.$$

So $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Therefore W is a vector space as every span in $\mathbb{R}^4$ is a subspace and therefore a vector space by a theorem from **Lecture 14**.

Alternative: We can rewrite the two equations that determine W into

$$\begin{aligned} a + 3b - c &= 0 \\ a + b + c - d &= 0 \end{aligned} \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}}_{=A} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $W = \text{Nul}(A)$. The nullspace of A is a subspace of $\mathbb{R}^4$ and therefore itself a vector space. So W is a vector space.

3. §4.2.16 Show that the set W below is the colum space $\text{Col}(A)$ of a matrix A.

$$W = \left\{ \begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix} \in \mathbb{R}^4, \text{ where } b, c, d \in \mathbb{R} \right\}$$

Solution: Regrouping the variables we see that

$$W = \left\{ b \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}, b, c, d \in \mathbb{R} \right\} = \left\{ \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}}_{=A} \begin{bmatrix} b \\ c \\ d \end{bmatrix}, b, c, d \in \mathbb{R} \right\}.$$

But that means that $W = \text{Col}(A)$.

Part B

4. §4.2.26 a,b,c,d,e. In the following A denotes an $m \times n$ matrix. Mark each statement as true or false. Justify your answer.

   a) A null space is a vector space.
   True If $T : V \to W$ is a linear map then $\text{Nul}(T)$ is a subspace of V. Any subspace is a vector space. Therefore $\text{Nul}(T)$ is a vector space.

   b) The column space $\text{Col}(A)$ of an $m \times n$ matrix A is in $\mathbb{R}^m$.
   True The column space is a subspace of the codomain, if $T(\mathbf{x}) = A\mathbf{x}$, then the codomain is $\mathbb{R}^m$.

   c) $\text{Col}(A)$ is the set of all solutions $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$.
   False The column space is in the codomain, not in the domain where the solutions of $A\mathbf{x} = \mathbf{b}$ are.

d) $\text{Nul}(A)$ is the kernel $\text{Ker}(T)$ of the mapping $T(\mathbf{x}) = A\mathbf{x}$.
True By the definition of a linear transformation

$$\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}\} = \text{Nul}(A).$$

e) The range $T(V)$ of a linear transformation $T : V \to W$ is a vector space.
True We have seen in the lecture that $T(V)$ is a subspace, therefore it is a vector space.

5. §4.2.32 Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}.$$

Find polynomials p_1 and p_2 that span $\text{Ker}(T) = \text{Nul}(T)$ and describe the range $T(\mathbb{P}_2)$.

Solution: We know that

$$\text{Ker}(T) = \{p \in \mathbb{P}_2, \text{ such that } T(p) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}. \text{ However, } T(p) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} p(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow p(0) = 0.$$

A polynomial in $\mathbb{P}_2$ can be written as $p(x) = a_0 + a_1 x + a_2 x^2$. Therefore

$$p(0) = 0 \Leftrightarrow a_0 + a_1 0 + a_2 0^2 = 0 \Leftrightarrow a_0 = 0.$$

This means that $p \in \text{Ker}(T) \Leftrightarrow p(x) = \boxed{a_1} x + \boxed{a_2} x^2$. This means that the kernel can be spanned by the two polynomials $p_1(x) = x$ and $p_2(x) = x^2$ or $\text{Ker}(T) = \text{Span}\{x, x^2\}$.
The range of T is the line

$$L = \{\begin{bmatrix} c \\ c \end{bmatrix}, c \in \mathbb{R}\}.$$

Clearly, for any polynomial p, we have $T(p) \in L$. Furthermore Let $p_c(x) = c$ be the polynomial which is constant c for all $x \in \mathbb{R}$. Then $T(p_c) = \begin{bmatrix} c \\ c \end{bmatrix}$. So every point on the line L is an image of a polynomial under T. This means $T(\mathbb{P}_2) = L$.

6. §4.3.14 We know that the matrix A is row-equivalent to the matrix B, where

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \boxed{1} & 2 & 0 & 4 & 5 \\ 0 & 0 & \boxed{5} & -7 & 8 \\ 0 & 0 & 0 & 0 & \boxed{-9} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{ef}(A)$$

Solution: We see that B is in echelon form. To find a basis of $\text{Nul}(A)$ we complete the row reduction of $B = \text{ef}(A)$ and find

$$\text{ref}(A) = \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & \boxed{1} & \frac{-7}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the reduced echelon form we read out the solution. We obtain:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \cdot \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=\mathbf{v}_1} + x_4 \cdot \underbrace{\begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}}_{=\mathbf{v}_1}, x_2, x_4 \in \mathbb{R}.$$

Therefore $\mathrm{Nul}(A) = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^5$.
We furthermore know that $\mathrm{Col}(A)$ is spanned by the **pivot columns of** A. Therefore

$$\mathrm{Col}(A) = \mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -5 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix}\} \subset \mathbb{R}^4.$$

Part C

7. §4.3.18 **(C)** Find a basis B for $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.
Note: For this problem you **should** use a computer algebra program like Wolfram Alpha to find the solutions. Unlike in the other problems you do not have to show your steps.

Solution: To find a basis of $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ we write the vectors as columns of the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5]$ and reduce the matrix to echelon form. Reducing to echelon form, we see that the pivot columns are 1,2 and 4. Therefore

$$\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\},$$

and the vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ are linearly independent and therefore form a basis of the span.

8. §4.3.34 Consider the polynomials

$$p_1(t) = 1 + t, p_2(t) = 1 - t \text{ and } p_3(t) = 2.$$

Find a linear dependence relation for these three polynomials, then find a basis B for $H = \mathrm{Span}\{p_1, p_2, p_3\}$.
Solution: We have, for example, that

$$1 \cdot p_1 + 1 \cdot p_2 + (-1) \cdot p_3 = \mathbf{0}.$$

We can take out any of these three polynomials and the remaining two will still span H. For the remaining two we have that they are not multiples of each other, so they are linearly independent and therefore form a basis.
Example:

$$H = \mathrm{Span}\{p_1, p_2, p_3\} = \mathrm{Span}\{p_1, p_2\},$$

and $B = \{p_1, p_2\}$ is a basis of H.

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9. §4.4.2 Find the vector $\mathbf{x}$ given in coordinates $[\mathbf{x}]_B$ with respect to the basis B.

$$B = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, [\mathbf{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.$$

Solution: The B-coordinates are the weights of the basis vectors in B in the linear combination. So

$$\mathbf{x} = \boxed{8} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \boxed{(-5)} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

10. §4.4.8 Find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to the basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$:

$$\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Solution: The B-coordinates $[\mathbf{x}]_B$ are the weights c_1, c_2 and c_3, such that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{x}.$$

We can find the B-coordinates by solving the system of equations with augmented matrix $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 | \mathbf{x}]$. We find

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

Part D

11. §4.4.10 Find the change-of-coordinates matrix P_B from B to the standard basis in $\mathbb{R}^3$, where

$$B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}$$

Solution: The change-of-coordinates matrix P_B, such that $\mathbf{x} = P_B[x]_B$ is the matrix

$$P_B = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3] = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}.$$

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12. §4.4.20 Suppose $W = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly dependent spanning set for a vector space V. Show that each $\mathbf{w} \in V$ can be expressed in more than one way as a linear combination of the vectors in W.

    Solution: We know that the vectors in W span V, so we know that for any $\mathbf{w} \in V$:

    $$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \ \ (1)\ .$$

    As the vectors are linearly dependent, we know that $d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4 = \mathbf{0}$ and at least one of the d_i is not zero. But then

    $$\mathbf{w} = \mathbf{w} + \mathbf{0} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + (c_3 + d_3)\mathbf{v}_3 + (c_4 + d_4)\mathbf{v}_4 \ \ (2)\ .$$

    We furthermore know that at least one of the d_i is not zero. so the linear combination in (2) is indeed different from the one in (1).

13. §4.4.32 Consider the three polynomials in $\mathbb{P}_2$:

    $$p_1(t) = 1 + t^2\ ,\ p_2(t) = t - 3t^2\ ,\ p_3(t) = 1 + t - 3t^2.$$

    a) Use coordinate vectors to show that these polynomials form a basis B of $\mathbb{P}_2$.

    Solution: In coordinates of the standard basis $E = \{1, t, t^2\}$ of $\mathbb{P}_2$ we have that

    $$[p_1]_E = \begin{bmatrix}1\\0\\1\end{bmatrix}\ ,\ [p_2]_E = \begin{bmatrix}0\\1\\-3\end{bmatrix} \text{ and } [p_3]_E = \begin{bmatrix}1\\1\\-3\end{bmatrix}.$$

    The polynoials are linearly independent and span $\mathbb{P}_2$ if and only if the coordinate vectors are linearly independent and span $\mathbb{R}^3$. This means the matrix

    $$P = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -3 & -3\end{bmatrix} \to \begin{bmatrix}\boxed{1} & 0 & 1\\0 & \boxed{1} & 1\\0 & 0 & \boxed{-1}\end{bmatrix} = \text{ef}(P).$$

    has a pivot in every row and column. We see from the echelon form $\text{ef}(P)$ of P that this is indeed the case. so the polynomials form a basis of $\mathbb{P}_2$.
    Note: We can also solve this problem "by hand" i.e. by solving linear equations for polynomials.

    b) Find $q \in \mathbb{P}_2$, such that $[q]_B = \begin{bmatrix}-1\\1\\2\end{bmatrix}$.

    Solution: We have that $[q]_E = P[q]_B$. Therefore

    $$[q]_E = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -3 & -3\end{bmatrix}\begin{bmatrix}-1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\3\\-10\end{bmatrix}. \text{ That means that } q(t) = 1 + 3t - 10t^2.$$