Math 22: Linear Algebra
Fall 2019 - Homework 6

Total: 20 points — Return date: Wednesday 10/30/19

Part A

1. Let V, W be vector spaces and $T : V \to W$ be a linear map that is both one-to-one and onto. You may assume that we know that in this case there is also an inverse map $T^{-1} : W \to V$ that is both one-to-one and onto and such that

$$T \circ T^{-1}(\mathbf{w}) = \mathbf{w} \text{ for all } \mathbf{w} \in W \text{ and } T^{-1} \circ T(\mathbf{v}) = \mathbf{v} \text{ for all } \mathbf{v} \in V.$$

a) Show that T^{-1} is also a linear map.
Solution: We have to show that for all $\mathbf{w}_1, \mathbf{w}_2 \in W$ and $c \in \mathbb{R}$:
1.) $T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2)$.
2.) $T^{-1}(c\mathbf{w}_1) = cT^{-1}(\mathbf{w}_1)$.
We start with 1.). As T is one-to-one and onto there are unique $\mathbf{v}_1, \mathbf{v}_2 \in V$, such that $\mathbf{w}_1 = T(\mathbf{v}_1)$ and $\mathbf{w}_2 = T(\mathbf{v}_2)$. So 1.) can be written as

$$\begin{aligned}
& T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2) \\
\Leftrightarrow\quad & T^{-1}(T(\mathbf{v}_1) + T(\mathbf{v}_2)) = T^{-1}(T(\mathbf{v}_1) + T^{-1}(T(\mathbf{v}_2)) \\
\overset{T^{-1}\circ T = Id}{\Leftrightarrow}\quad & T^{-1}(T(\mathbf{v}_1) + T(\mathbf{v}_2)) = \mathbf{v}_1 + \mathbf{v}_2 \\
\overset{T \text{ is a linear map}}{\Leftrightarrow}\quad & T^{-1}(T(\mathbf{v}_1 + \mathbf{v}_2)) = \mathbf{v}_1 + \mathbf{v}_2 \\
\overset{T^{-1}\circ T = Id}{\Leftrightarrow}\quad & \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_2.
\end{aligned}$$

Now the last statement is true. As all the steps are reversible, we know that also the first equation must be true. So 1.) is true for T^{-1}.
Now we look at 2.). As T is one-to-one and onto there are unique $\mathbf{v}_1 \in V$, such that $\mathbf{w}_1 = T(\mathbf{v}_1)$. So 2.) can be written as

$$\begin{aligned}
& T^{-1}(c\mathbf{w}_1) = cT^{-1}(\mathbf{w}_1) \\
\Leftrightarrow\quad & T^{-1}(cT(\mathbf{v}_1)) = cT^{-1}(T(\mathbf{v}_1)) \\
\overset{T^{-1}\circ T = Id}{\Leftrightarrow}\quad & T^{-1}(cT(\mathbf{v}_1)) = c\mathbf{v}_1 \\
\overset{T \text{ is a linear map}}{\Leftrightarrow}\quad & T^{-1}(T(c\mathbf{v}_1)) = c\mathbf{v}_1 \\
\overset{T^{-1}\circ T = Id}{\Leftrightarrow}\quad & c\mathbf{v}_1 = c\mathbf{v}_1.
\end{aligned}$$

Now the last statement is true. As all the steps are reversible, we know that also the first equation must be true. So 2.) is true for T^{-1}. In total we have shown that T^{-1} is a linear map.

b)* **(optional)** Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent in V if and only if the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$ are linearly independent in W.
Solution: We have to show both directions. We start with the first direction:

$$(1)\quad \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \text{ lin. independent in } V \Rightarrow T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p) \text{ lin. independent in } W \quad (2).$$

We know that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent, so

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \text{ implies } c_1 = c_2 = \ldots = c_p = 0. \quad (3)$$

We want to show that the same is true for the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$. Suppose that

$$d_1T(\mathbf{v}_1) + d_2T(\mathbf{v}_2) + \ldots + d_pT(\mathbf{v}_p) = \mathbf{0} \overset{T \text{ linear}}{\Leftrightarrow} T(\underbrace{d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_p\mathbf{v}_p}_{=\mathbf{v}}) = \mathbf{0}.$$

But T is one-to-one, so if $T(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$. So

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_p\mathbf{v}_p = \mathbf{v} = \mathbf{0}.$$

But by (3) that means that $d_1 = d_2 = \ldots = d_p = 0$. But these were the weights for the vectors $(T(\mathbf{v}_i))_i$. So these vectors are also linearly independent.

(2) $\Rightarrow$ (1) We now prove the inverse direction. So

$$d_1T(\mathbf{v}_1) + d_2T(\mathbf{v}_2) + \ldots + d_pT(\mathbf{v}_p) = \mathbf{0} \Rightarrow d_1 = d_2 = \ldots = d_p = 0.$$

We now write down a linear dependence relation for the vectors $(\mathbf{v}_i)_i$ and take T on both sides:

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \quad &\Rightarrow \quad T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p) = T(\mathbf{0}) \\ &\overset{T \text{ linear}}{\Leftrightarrow} \quad c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0}. \end{aligned}$$

As the vectors $(T(\mathbf{v}_i))_i$ are linearly independent, this implies that $c_1 = c_2 = \ldots = c_p = 0$. This means that the vectors $(\mathbf{v}_i)_i$ are also linearly independent.

c) Show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ span V if and only if the vectors $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_r)$ span W.
Solution: We first show the direction

$$(1) \quad \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r) = V \Rightarrow \text{Span}(T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_r)) = W. \quad (2)$$

Take any $\mathbf{w}$ in W. We know that there is $\mathbf{v} \in V$, such that $\mathbf{w} = T(\mathbf{v})$ as T is onto. Furthermore for this $\mathbf{v} \in V$ we know that there are weights $c_1, c_2, \ldots, c_r$, such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_r\mathbf{u}_r.$$

Taking T on both sides, we obtain:

$$\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_r\mathbf{u}_r) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \ldots + c_rT(\mathbf{u}_r).$$

So any $\mathbf{w} \in W$ is in the span of the $(T(\mathbf{v}_i))_i$.
(2) $\Rightarrow$ (1) We now prove the inverse direction. So we take any $\mathbf{v}$ in V. We know that there

is $\mathbf{w} \in W$, such that $\mathbf{v} = T^{-1}(\mathbf{w})$ as T^{-1} is onto. Furthermore for this $\mathbf{w} \in W$ we know that there are weights $c_1, c_2, \ldots, c_r$, such that

$$\mathbf{w} = c_1 T(\mathbf{u}_1) + c_2 T(\mathbf{u}_2) + \ldots + c_r T(\mathbf{u}_r) = T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_r\mathbf{u}_r).$$

Taking T^{-1} on both sides, we obtain:

$$\mathbf{v} = T^{-1}(\mathbf{w}) = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_r\mathbf{u}_r.$$

So any $\mathbf{v} \in V$ is in the span of the $(\mathbf{v}_i)_i$.
In total we have shown

$$(1) \quad \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r) = V \Leftrightarrow \text{Span}(T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_r)) = W. \quad (2)$$

d) Argue that analogous statements for b) and c) are true for T^{-1}.
Solution: All we used was that T is linear and both one-to-one and onto. As the same is true for T^{-1} analogous statements for b) and c) apply to T^{-1}.

Note: Let V be a vector space with basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and

$$T : V \to \mathbb{R}^n, \mathbf{v} \mapsto T(\mathbf{v}) = [\mathbf{v}]_B$$

be the coordinate map. Then this theorem says that linear dependence / independence and spanning of vectors can always be examined in coordinates.

2. §4.5.12 Find the dimension of the subspace H given by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix} \text{ and } \mathbf{v}_4 = \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}.$$

Solution: Following the lecture, we put the vectors into a matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$ and row reduce the matrix to find the pivot columns.

$$A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] = \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & -3 & -8 & -3 \\ 0 & \boxed{1} & 5 & 7 \\ 0 & 0 & 0 & \boxed{4} \end{bmatrix} = \text{ef}(A).$$

We find that there are three pivot columns so the dimension of the subspace H is three. A basis B of H is $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$.

3. §4.5.22 The first four Laguerre polynomials are

$$p_1(t) = 1 \ , \ p_2(t) = 1 - t \ , \ p_3(t) = 2 - 4t + t^2 \ \text{ and } \ p_4(t) = 6 - 18t + 9t^2 - t^3.$$

Show that these polynomials form a basis B of $\mathbb{P}_3$.
Solution: We know that $\dim(\mathbb{P}_3) = 4$. The coordinate vectors in terms of the standard basis $E = \{1, t, t^2, t^3\}$ are

$$[p_1]_E = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \ [p_2]_E = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \ , \ [p_3]_E = \begin{bmatrix}2\\-4\\1\\0\end{bmatrix} \text{ and } [p_4]_E = \begin{bmatrix}6\\-18\\9\\-1\end{bmatrix}.$$

Putting the vectors into a matrix $A = [[p_1]_E, [p_2]_E, [p_3]_E, [p_4]_E]$ we see that

$$A = \begin{bmatrix}\boxed{1} & 1 & 2 & 6\\ 0 & \boxed{-1} & -4 & -18\\ 0 & 0 & \boxed{1} & 9\\ 0 & 0 & 0 & \boxed{-1}\end{bmatrix}.$$

A is already in echelon form and has a pivot in every column and row. That means that $B = \{p_1, p_2, p_3, p_4\}$ is a basis of $\mathbb{P}_3$.
Note: It is already sufficient to show that the polynomials are either linearly independent or spanning $\mathbb{P}_3$. As there are 4 and the dimension of $\mathbb{P}_3$ is also 4, that implies that these polynomials form a basis.

4. §4.5.30 Mark each statement as true or false. Justify your answer.

a) If there exists a linearly dependent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V, then $\dim(V) \leq p$.
False A counterexample is the set $\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\3\\3\end{bmatrix} \}$ in $\mathbb{R}^3$. Then $\dim(\mathbb{R}^3) = 3$ which is greater than 2.

b) If every set of p vectors in V fails to span V, then $\dim(V) > p$.
True If $\dim(V) \leq p$ then there would be a basis of V with p or less elements. This basis would span V, which contradicts the statement. So $\dim(V) > p$.

c) If $p \geq 2$, and $\dim(V) = p$ then every set of $p - 1$ non-zero vectors is linearly independent.
False For a counterexample, consider the set in part a).

Part C

5. §4.6.2 Assume that the matrices A and B are row equivalent, where

$$A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5] = \begin{bmatrix}1 & -3 & 4 & -1 & 9\\ -2 & 6 & -6 & -1 & -10\\ -3 & 9 & -6 & -6 & -3\\ 3 & -9 & 4 & 9 & 0\end{bmatrix} \text{ and } B = \begin{bmatrix}\boxed{1} & -3 & 0 & 5 & -7\\ 0 & 0 & \boxed{2} & -3 & 8\\ 0 & 0 & 0 & 0 & \boxed{5}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\\ \mathbf{r}_4\end{bmatrix}.$$

a) List $\text{rank}(A)$ and $\dim(\text{Nul}(A))$.
Solution: $\text{rank}(A) = \#$pivot columns $= 3$ and $\dim(\text{Nul}(A) = \#$non-pivot columns $= 2$.

b) Find a basis of $\text{Col}(A)$.
Solution: A basis B_c of $\text{Col}(A)$ is formed by the pivot columns of $\boxed{A}$. So $B_c = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$.

c) Find a basis of $\text{Row}(A)$.
Solution: A basis B_r for $\text{Row}(A) = \text{Col}(A^T)$ is formed by the pivot rows of B, so $B_r = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$.

d) Find a basis of $\text{Nul}(A)$.
Solution: To find a basis for $\text{Nul}(A)$ we have to solve the equation $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$. To this end we reduce $[B|\mathbf{0}]$ (or $[A|\mathbf{0}]$) to echelon form and write the solutions in parametric vector form:

$$[B|\mathbf{0}] = \left[\begin{array}{ccccc:c} \boxed{1} & -3 & 0 & 5 & -7 & 0 \\ 0 & 0 & \boxed{2} & -3 & 8 & 0 \\ 0 & 0 & 0 & 0 & \boxed{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc:c} \boxed{1} & -3 & 0 & 5 & 0 & 0 \\ 0 & 0 & \boxed{1} & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Writing out the corresponding equations and expressing the basic variables in terms of free variables we obtain:

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=\mathbf{v}_1} + x_4 \underbrace{\begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}}_{=\mathbf{v}_2}, \text{ where } x_2, x_4 \in \mathbb{R}.$$

A basis B_n of $\text{Nul}(A)$ is $B_n = \{\mathbf{v}_1, \mathbf{v}_2\}$.

6. §4.6.8 Suppose a 5×6 matrix A has four pivot columns. What is $\dim(\text{Nul}(A))$? Is $\text{Col}(A) = \mathbb{R}^4$? Why or why not?
Solution: We know that $\dim(\text{Col}(A)) = 4$ as the dimension of the column space is equal to the number of pivot columns. As $\dim(\text{Nul}(A)) + \dim(\text{Col}(A)) = 6$ we know that $\dim(\text{Nul}(A)) = 2$. It is **not** true that $\text{Col}(A)$ is $\mathbb{R}^4$. Though $\dim(\text{Col}(A)) = 4$ it is a subspace of $\mathbb{R}^5$.

7. §4.6.18 Mark each statement as true or false. Justify your answer.

a) If B is any echelon form of A, then the pivot columns of B form a basis of the column space of A.
False Only the pivot columns of A span the column space of A.

b) Row operations preserve the linear dependence relations among the rows of A.
False Considering, for example, a row swap, we see that the weights change, so linear dependence relations are not preserved. However, the row operations preserve the row space.

c) The dimension of $\mathrm{Nul}(A)$ is the number of columns of A that are not pivot columns.
True The number of non-pivot columns is equal to the number of free variables. The number of free variables is equal to the number of basis vectors of $\mathrm{Nul}(A)$, which is the dimension of $\mathrm{Nul}(A)$.

d) The row space of A^T is the same as the column space of A.
True This is true as the columns of A are the rows of A^T.

e) If A and B are row equivalent, then their row spaces are the same.
True We saw in the lecture that row operations do not change the row space.

8. §4.6.26 Explain why an $m \times n$ matrix A, where $m > n$ has full rank if and only if its columns are linearly independent.
Solution: As $m > n$ and $\mathrm{rank}(A) \leq \min\{m, n\}$ we know that $\mathrm{rank}(A) \leq n$. Therefore it is maximal, if $\mathrm{rank}(A) = n$ or if A has n pivots. This is equal to the condition that the columns of A are linearly independent.

Part D

9. §4.7.6 Let $D = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be two bases for a vector space V and suppose that

$$\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3 \ , \ \mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3 \ , \ \mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3.$$

a) Find the change of coordinate matrix from F to D.
Solution: To find the change-of-coordinates matrix $\underset{D\leftarrow F}{P}$ we first determine the coordinates vectors of $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ in D coordinates. We have

$$[\mathbf{f}_1]_D = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \ , \ [\mathbf{f}_2]_D = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \ , \ [\mathbf{f}_3]_D = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}. \text{ So we have } \underset{D\leftarrow F}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

b) Find $[\mathbf{x}]_D$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.
Solution: We know that

$$[\mathbf{x}]_F = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \text{ so } [\mathbf{x}]_D = \underset{D\leftarrow F}{P}[\mathbf{x}]_F = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}.$$

10. §4.7.10 For the bases $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ find the change of coordinate matrices $\underset{C\leftarrow B}{P}$ and $\underset{B\leftarrow C}{P}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \ , \ \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \ , \ \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Solution: Let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, we know that

$$P_B = \underset{E\leftarrow B}{P} = [\mathbf{b}_1, \mathbf{b}_2] = \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} \text{ and } P_C = \underset{E\leftarrow C}{P} = [\mathbf{c}_1, \mathbf{c}_2] = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}.$$

Then

$$\underset{C\leftarrow B}{P} = \underset{C\leftarrow E}{P}\cdot\underset{E\leftarrow B}{P} = \underset{E\leftarrow C}{P^{-1}}\cdot\underset{E\leftarrow B}{P} = P_C^{-1}\cdot P_B = \frac{1}{3}\begin{bmatrix}2 & -5\\ -1 & 4\end{bmatrix}\cdot\begin{bmatrix}7 & 2\\ -2 & -1\end{bmatrix} = \begin{bmatrix}8 & 3\\ -5 & -2\end{bmatrix}.$$

$$\underset{B\leftarrow C}{P} = \underset{B\leftarrow C}{P^{-1}} = \begin{bmatrix}2 & 3\\ -5 & -8\end{bmatrix}.$$

11. §4.7.14 In $\mathbb{P}_2$ find the change of coordinate matrix P_B from the basis

$$B = \{p_1, p_2, p_3\} = \{1-3t^2\ ,\ 2+t-5t^2\ ,\ 1+2t\} \text{ to } E = \{1\ ,\ t\ ,\ t^2\}.$$

Then write t^2 as a linear combination of the polynomials in B.
Solution: To find the change-of-coordinates matrix $\underset{E\leftarrow B}{P} = P_B$ we first determine the coordinates vectors of p_1, p_2, p_3 in E coordinates.

$$[p_1]_E = \begin{bmatrix}1\\0\\-3\end{bmatrix}\ ,[p_2]_E = \begin{bmatrix}2\\1\\-5\end{bmatrix}\ ,[p_3]_E = \begin{bmatrix}1\\2\\0\end{bmatrix}\ ,\text{ Therefore } P_B = \begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 2\\ -3 & -5 & 0\end{bmatrix}.$$

We know that $[t^2]_E = \begin{bmatrix}0\\0\\1\end{bmatrix} = P_B\cdot[t^2]_B$, so $P_B^{-1}[t^2]_E = [t^2]_B$ and $[t^2]_B = \begin{bmatrix}3\\-2\\1\end{bmatrix}$.

12. §4.7.19 **(C)** Let

$$P = \begin{bmatrix}1 & 2 & -1\\ -3 & -5 & 0\\ 4 & 6 & 1\end{bmatrix} \text{ and } \mathbf{v}_1 = \begin{bmatrix}-2\\2\\3\end{bmatrix}\ ,\ \mathbf{v}_2 = \begin{bmatrix}-8\\5\\2\end{bmatrix}\ ,\ \mathbf{v}_3 = \begin{bmatrix}-7\\2\\6\end{bmatrix}.$$

Note: For this problem you **should** use a computer algebra program like Wolfram Alpha to find the solutions. Unlike in the other problems you do not have to show your steps.

a) Find a basis $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$, such that $P = \underset{V\leftarrow U}{P}$ is the change-of-coordinates matrix from U to V.
Solution: We know that $P_V = \underset{E\leftarrow V}{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is the change-of-coordinates matrix from V to the standard basis E. We also know that

$$\underset{V\leftarrow U}{P} = \underset{V\leftarrow E}{P}\cdot\underset{E\leftarrow U}{P} = P_V^{-1}\cdot P_U.\ \text{Therefore } P_U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = P_V\cdot\underset{V\leftarrow U}{P} = \begin{bmatrix}-6 & -6 & -5\\ -5 & -9 & 0\\ 21 & 32 & 3\end{bmatrix}.$$

b) Find a basis $W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, such that $P = \underset{W\leftarrow V}{P}$ is the change-of-coordinates matrix from V to W.
Solution: In a similar fashion we have

$$\underset{W\leftarrow V}{P} = \underset{W\leftarrow E}{P}\cdot\underset{E\leftarrow V}{P} = P_W^{-1}\cdot P_V.\ \text{So } P_W = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = P_V\cdot\underset{W\leftarrow V}{P^{-1}} = \begin{bmatrix}28 & 38 & 21\\ -9 & -13 & -7\\ -3 & 3 & 2\end{bmatrix}.$$