Math 22: Linear Algebra
Fall 2019 - Homework 7

Total: 20 points Return date: Wednesday 11/06/19

Numbered problems are taken from Lay, D. et al: *Linear Algebra with Applications*, fifth edition.
Please show your work; no credit is given for solutions without work or justification.

Part A

1. §6.1.14 Find the distance between

$$\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}.$$

Solution: We have $\text{dist}(\mathbf{u}, \mathbf{z}) = \|\mathbf{u} - \mathbf{z}\|$. So

$$\text{dist}(\mathbf{u}, \mathbf{z}) = \| \begin{bmatrix} -4 \\ -4 \\ -6 \end{bmatrix} \| = ((-4)^2 + (-4)^2 + (-6)^2)^{1/2} = \sqrt{68}.$$

2. §6.1.18 Determine whether the vectors $\mathbf{y}$ and $\mathbf{z}$ are orthogonal.

Solution: Two vectors are orthogonal if the dot product is zero. So we check:

$$\mathbf{y} \bullet \mathbf{z} = \mathbf{y}^T \cdot \mathbf{z} = \begin{bmatrix} -3 & 7 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = (-3) \cdot 1 + 7 \cdot (-8) + 4 \cdot 15 + 0 \cdot (-7) = 1 \neq 0.$$

So the two vectors are not orthogonal.

3. §6.1.30 Let W be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors orthogonal to W, i.e.

$$W^\perp = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{w}^T \cdot \mathbf{x} = 0, \text{ for all } \mathbf{w} \in W\}.$$

Show that $W^\perp$ is a subspace of of $\mathbb{R}^n$ in the proposed steps.

Solution:

a) We first show that if $\mathbf{u} \in W^\perp$ and c a number, then $c\mathbf{u} \in W^\perp$.
Take any $\mathbf{w} \in W$. We know that $\mathbf{w}^T \cdot \mathbf{u} = 0$. But then

$$\mathbf{w}^T \cdot (c\mathbf{u}) = c(\mathbf{w}^T \cdot \mathbf{u}) = c \cdot 0 = 0.$$

As $\mathbf{w}$ was chosen arbitrarily this is true for any $\mathbf{w} \in W$. So $c\mathbf{u} \in W^\perp$.

b) We then show that if $\mathbf{u}, \mathbf{v} \in W^{\perp}$, then $c\mathbf{u} + \mathbf{v} \in W^{\perp}$.
Take any $\mathbf{w} \in W$. We know that $\mathbf{w}^T \cdot \mathbf{u} = 0$ and $\mathbf{w}^T \cdot \mathbf{v} = 0$. But then

$$\mathbf{w}^T \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w}^T \cdot \mathbf{u} + \mathbf{w}^T \cdot \mathbf{v} = 0 + 0 = 0.$$

As $\mathbf{w}$ was chosen arbitrarily this is true for any $\mathbf{w} \in W$. So $\mathbf{u} + \mathbf{v} \in W^{\perp}$.

c) Finally we have to show that $\mathbf{0} \in W^{\perp}$. But $\mathbf{w}^T \cdot \mathbf{0} = 0$ for any $\mathbf{w} \in W$. So $\mathbf{0} \in W^{\perp}$.

In total we have that $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Part B

4. Let A be an $m \times n$ matrix. We want to see an important relationship between the row space $\text{Row}(A)$ and the null space $\text{Nul}(A)$ of A.

a) Show that $\text{Row}(A)^{\perp} = \text{Nul}(A)$.
Hint: Show that if $\mathbf{x}$ is in $\text{Nul}(A)$, then it is orthogonal to each row of A.

Solution: Let A be the matrix with colum vectors $(\mathbf{a}_i)_i$ and row vectors $(\mathbf{r}_j)_j$. Let $\bullet$ denote the dot product. Then

$$A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}. \text{ Then } A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \bullet \mathbf{x} \\ \mathbf{r}_2 \bullet \mathbf{x} \\ \vdots \\ \mathbf{r}_m \bullet \mathbf{x} \end{bmatrix}.$$

Take $\mathbf{x} \in \text{Nul}(A)$. Then $A\mathbf{x} = \mathbf{0}$ implies that

$$\mathbf{r}_1 \bullet \mathbf{x} = \mathbf{r}_2 \bullet \mathbf{x} = \ldots = \mathbf{r}_m \bullet \mathbf{x} = 0.$$

So $\mathbf{x}$ is orthogonal to every row of A. But the rows of A span $\text{Row}(A)$, so by **exercise 3** we know that $\mathbf{x}$ is orthogonal to every vector in $\text{Row}(A)$. So $\mathbf{x}$ in $\text{Row}(A)^{\perp}$. As $\mathbf{x}$ was chosen arbitrarily , we have $\text{Nul}(A) \subset \text{Row}(A)^{\perp}$.
Conversely if $\mathbf{y} \in \text{Row}(A)^{\perp}$, then $\mathbf{y}$ is orthogonal to every row of A. That implies that $\mathbf{y}$ in $\text{Nul}(A)$. So $\text{Row}(A)^{\perp} \subset \text{Nul}(A)$. In total: $\text{Row}(A)^{\perp} = \text{Nul}(A)$.

b) Show that $\text{Col}(A)^{\perp} = \text{Nul}(A^T)$. **Hint:** Use part a).

Solution: We know that $\text{Col}(A) = \text{Row}(A^T)$. So

$$\text{Col}(A)^{\perp} = \text{Row}(A^T)^{\perp} = \text{Nul}(A^T).$$

c) Let $W = \text{Span}\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\4\end{bmatrix}\}$. Find $W^\perp$ and determine the dimensions of W and $W^\perp$.

Solution: Following part b) we put the vectors spanning W into a matrix

$$A = \begin{bmatrix}1 & 2\\2 & 4\\1 & 4\end{bmatrix}. \text{ Then } \text{Span}\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\4\end{bmatrix}\} = \text{Col}(A).$$

So we have to find $\text{Nul}(A^T)$. To this end we solve the linear equations with augmented matrix $[A^T|\mathbf{0}]$. Using row reduction we obtain

$$\text{Nul}(A^T) = \text{Span}\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\} = \text{Col}(A)^\perp = W^\perp.$$

5. §6.2.6 Check if the vectors

$$\mathbf{v}_1 = \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix}3\\3\\5\\-1\end{bmatrix}$$

are orthogonal.

Solution: We have that $\mathbf{v}_2 \bullet \mathbf{v}_3 = (-12) + 3 + (-15) + (-8) = -32 \neq 0$ we know that the vectors are not orthogonal.

6. §6.2.8 Show that $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis of $\mathbb{R}^2$ and the express $\mathbf{x}$ as a linear combination of the vectors in B.

$$\mathbf{u}_1 = \begin{bmatrix}3\\1\end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix}-2\\6\end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix}-6\\3\end{bmatrix}.$$

Solution: It is easy to check that $\mathbf{u}_1 \bullet \mathbf{u}_2 = 0$. As we have two vectors, by the lecture this implies that B is a basis. Additionally by the lecture we know that

$$\mathbf{x} = \frac{\mathbf{x} \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1} \cdot \mathbf{u}_1 + \frac{\mathbf{x} \bullet \mathbf{u}_2}{\mathbf{u}_2 \bullet \mathbf{u}_2} \cdot \mathbf{u}_2. \text{ So } \mathbf{x} = -\frac{3}{2}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2.$$

Part C

7. §6.2.20 Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors

$$\mathbf{u} = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix} \text{ and } \mathbf{v} = \frac{1}{3}\begin{bmatrix}1\\2\\0\end{bmatrix}.$$

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Check if the vectors are orthogonal. If they are not orthonormal, normalize the vectors to create an orthonormal set.

Solution: We have that $\mathbf{u} \bullet \mathbf{v} = 0$. That means that these two vectors are orthogonal. Furthermore

$$\|\mathbf{u}\| = \|\frac{1}{3} \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \| = \frac{1}{3} \cdot 3 = 1 \ , \ \|\mathbf{v}\| = \|\frac{1}{3} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \| = \frac{\sqrt{5}}{3}.$$

Scaling the second vector to unit length we find $\|\tilde{\mathbf{v}}\| = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

8. §6.2.24 True / False questions.

   a) Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.
   Solution: This is true. If the zero vector is contained in the set then the set is orthogonal but not linearly independent. However, if the zero vector is not contained, the set is automatically linearly independent.

   b) If a set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \bullet \mathbf{u}_j = 0.$ whenever $i \neq j$, then S is an orthonormal set.
   Solution: This is false. To be orthonormal, the vectors have to be unit vectors.

   c) If the columns of an $m \times n$ matrix A are orthonormal, then the linear map $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.
   Solution: This is true. We have seen this in **Lecture 21, Theorem 7**.

   d) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.
   Solution: This is true. These two vectors span the same subspace. So the projection is the same.

   e) An orthogonal matrix is invertible.
   Solution: This is true. As the columns of an orthogonal matrix are linearly independent, the matrix is invertible.

Part D

9. §6.3.12 Find the closest point to $\mathbf{y}$ in the subspace W spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \ , \ \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Solution: Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. So the closest point to $\mathbf{y}$ in W is the projection $\text{proj}_W(\mathbf{y})$. The formula is

$$\text{proj}_W(\mathbf{y}) = \frac{\mathbf{y} \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1} \cdot \mathbf{v}_1 + \frac{\mathbf{y} \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2} \cdot \mathbf{v}_2. \text{ So } \text{proj}_W(\mathbf{y}) = 3\mathbf{v}_1 + 1\mathbf{v}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

10. §6.3.18 Let $\mathbf{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $W = \text{Span}\{\mathbf{u}_1\}$.

    a) Let $U = [\mathbf{u}_1]$. Compute UU^T and U^TU.
    Solution:

    $$UU^T = \frac{1}{10} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & -3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \text{ and } \frac{1}{10} \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \|\mathbf{u}_1\|^2 = 1.$$

    b) Compute $\text{proj}_W(\mathbf{y})$ and $(UU^T)\mathbf{y}$.
    Solution: We have that

    $$\text{proj}_W(\mathbf{y}) = \mathbf{y} \bullet \mathbf{u}_1 \cdot \mathbf{u}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} = (UU^T)\mathbf{y}.$$

11. §6.3.23 Let A be an $m \times n$ matrix. Prove that every vector $\mathbf{x} \in \mathbb{R}^n$ can be written in the form

$$\mathbf{x} = \mathbf{p} + \mathbf{u}, \text{ where } \mathbf{p} \in \text{Row}(A) \text{ and } \mathbf{u} \in \text{Nul}(A).$$

Then show that there is an unique $\mathbf{p} \in \text{Row}(A)$, such that $A\mathbf{p} = \mathbf{b}$.

Solution: We know from **exercise 4** that $\text{Row}(A)^\perp = \text{Nul}(A)$. That means that

$$\mathbb{R}^n = \text{Row}(A) + \text{Nul}(A). \text{ So } \mathbf{x} = \mathbf{p} + \mathbf{u} \text{ where } \mathbf{p} \in \text{Row}(A) \text{ and } \mathbf{u} \in \text{Nul}(A).$$

Now if $A\mathbf{x} = \mathbf{b}$ is consistent, then

$$A\mathbf{x} = A(\mathbf{p} + \mathbf{u}) = A\mathbf{p} + A\mathbf{u} = \mathbf{b} + \mathbf{0} = \mathbf{b}. \text{ So } A\mathbf{p} = \mathbf{b}.$$

That means that for all $\mathbf{b} \in \text{Col}(A)$, there is $\mathbf{p} \in \text{Row}(A)$, that maps onto $\mathbf{b}$.
Now the map $A : \text{Row}(A) \to \text{Col}(A), \mathbf{p} \mapsto A\mathbf{p}$ is onto. As $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$ it must also be one-to-one.