Math 22: Linear Algebra
Fall 2019 - Homework 8

Total: 20 points Return date: Wednesday 11/13/19

Numbered problems are taken from Lay, D. et al: *Linear Algebra with Applications*, fifth edition.
Please show your work; no credit is given for solutions without work or justification. For questions marked with a **(C)** a computer algebra system can be used and intermediate steps are not asked.

Part B

1. §6.4.10 Find an orthogonal basis for the column space $\mathrm{Col}(A)$ of the matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3].$$

Solution: To find an orthogonal basis $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ we have to apply the Gram-Schmidt process to th basis vectors $B = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ formed by the columns of the matrix A. Following the algorithm we get

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{a}_1 \\ \mathbf{u}_2 &= \mathbf{a}_2 - \frac{\mathbf{a}_2 \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \mathbf{a}_2 - (-3)\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \\ \mathbf{u}_3 &= \mathbf{a}_3 - \frac{\mathbf{a}_3 \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{a}_3 \bullet \mathbf{u}_2}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}. \end{aligned}$$

So the orthogonal basis is $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

2. §6.4.18 True / False questions

   a) If $W = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.
   Solution: This is false. The zero vector $\mathbf{0}$ could be in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. However, any set of non-zero orthogonal vectors is linearly independent.

   b) If $\mathbf{x}$ is not in a subspace W then $\mathbf{x} - \mathrm{proj}_W(\mathbf{x})$ is not zero.
   Solution: This is true. $\mathbf{z} = \mathbf{x} - \mathrm{proj}_W(\mathbf{x}) \in W^\perp$. By the **Orthogonal Decomposition Theorem** we have that $\mathbf{x} = \mathrm{proj}_W(\mathbf{x}) + \mathbf{z}$. So if $\mathbf{z} = \mathbf{0}$ then $\mathbf{x} \in W$, a contradiction.

   c) In a QR factorization, say $A = QR$ (when A has linearly dependent columns), the columns of Q form an orthonormal basis of $\mathrm{Col}(A)$.
   Solution: This is true. This follows from the lecture.

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3. §6.5.4 Find a least square solution $\hat{\mathbf{x}}$ of $A\hat{\mathbf{x}} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: Following the lecture we have to solve the equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. We obtain:

$$A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \quad A^T\mathbf{b} = \begin{bmatrix} 6 \\ 14 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Part C

4. §6.6.12 **(C)** A child's systolic blood pressure p and weight w are approximately related by the equation

$$\beta_0 + \beta_1 \ln(w) = p$$

Use the experimental data given in the text to estimate the child's blood pressure.
Solution: We have to find an (approximative) least square solution for the following equations:

$$\begin{array}{rcl} \beta_0 + \beta_1 \cdot 3.78 & = & 91 \\ \beta_0 + \beta_1 \cdot 4.11 & = & 98 \\ \beta_0 + \beta_1 \cdot 4.39 & = & 103 \\ \beta_0 + \beta_1 \cdot 4.73 & = & 110 \\ \beta_0 + \beta_1 \cdot 4.88 & = & 112 \end{array} \quad \text{or} \quad X\hat{\beta} = \begin{bmatrix} 1 & 3.78 \\ 1 & 4.11 \\ 1 & 4.39 \\ 1 & 4.73 \\ 1 & 4.88 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 91 \\ 98 \\ 103 \\ 110 \\ 112 \end{bmatrix} = \mathbf{y}.$$

So we have to solve the equation $X^T X\hat{\beta} = X^T\mathbf{y}$. We obtain

$$\hat{\beta} = \begin{bmatrix} 18.56 \\ 19.24 \end{bmatrix}. \quad \text{Therefore } p = 18.56 + 19.24\ln(w).$$

For $w = 100$ we obtain approximately $\boxed{p = 107}$.

5. §5.1.12 Find a basis for the eigenspaces of A, where

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \text{ and } \lambda_1 = 1 \text{ , } \lambda_2 = 5.$$

Solution: As we already know the eigenvalues we have to find the eigenspaces
$\text{Eig}(A, \lambda_i) = \text{Nul}(A - \lambda_i I_2)$. We obtain:

$$\begin{array}{rcl} \lambda_1 = 1 : A - I_2 & = & \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \text{ and } \text{Nul}(A - I_2) = \text{Span}\{\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}\} \\ \lambda_2 = 5 : A - 5I_2 & = & \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix} \text{ and } \text{Nul}(A - 5I_2) = \text{Span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}. \end{array}$$

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6. §5.1.20 Without calculation, find one eigenvalue and **two** linearly independent eigenvectors of

$$A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}.$$

Solution: The matrix is not invertible, as all rows are the same. So we know that

$$A\mathbf{x} = \mathbf{0} = 0\mathbf{x} \text{ has a non-trivial solution.}$$

It is easy to see that $\dim(A - 0I_3) = 2$. A solution is any vector $\mathbf{x}$, such that

$$x_1 \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can see that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are two linearly independent solutions.

Additionally, as the sum in each row is the same, we can also find

$$\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{=\mathbf{v}_3} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 15 \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{=\mathbf{v}_3}.$$

So $\mathbf{v}_3$ is an eigenvector for the eigenvalue 15.

Part D

7. §5.1.26 Show that if $A^2 = O$ is the zero matrix, then the only eigenvalue of A is 0.
Solution: If $\mathbf{x} \neq \mathbf{0}$ is an eigenvector, then

$$A\mathbf{x} = \lambda \cdot \mathbf{x}.$$

To relate this equation to $A^2 = A \cdot A = O$, we multiply both sides by A from the left and obtain

$$\mathbf{0} = O\mathbf{x} = A^2\mathbf{x} = A(\lambda \cdot \mathbf{x}) = \lambda \cdot (A\mathbf{x}) = \lambda^2\mathbf{x}. \text{ Therefore } \lambda^2\mathbf{x} = \mathbf{0}.$$

As $\mathbf{x} \neq \mathbf{0}$, we obtain $\lambda^2 = 0$ hence $\lambda = 0$.

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8. §5.2.12 Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution: We have to find $\det(A - \lambda I_3)$. We obtain

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \cdot \begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)(-1-\lambda)(4-\lambda) = -\lambda^3 + 5\lambda^2 - 2\lambda - 8. \end{aligned}$$

9. §5.2.18 Find h in the matrix A, such that the eigenspace for $\lambda = 5$ is two-dimensional, where

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution: For the matrix $A - 5I_4$ we obtain

$$A - 5I_4 = B = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

We can find the dimension of $\text{Nul}(A - 5I_4) = \text{Nul}(B)$ by row reducing $[B|\mathbf{0}]$ to echelon form. We find

$$[B|\mathbf{0}] = \left[\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & \boxed{1} & -3 & 0 & 0 \\ 0 & 0 & h-6 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The dimension of $\text{Nul}(B)$ is the number of non-pivot columns. We get two non-pivot columns if $\boxed{h = 6}$. So $\dim(\text{Nul}(A - 5I_4)) = 2$ if $h = 6$.