Math 22: Linear Algebra
Fall 2019 - Homework 9

not graded

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1. Consider the following linear transformations (see **Example 3** and **Tables 1 & 3** on pages 73 - 76 of the book).
   Determine whether the corresponding standard matrix of the given transformation has a real eigenvalue. If so, state **one** eigenvalue and a corresponding eigenvector. Justify your answer.

   a) $T : \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{v} \mapsto T(\mathbf{v}) = 3\mathbf{v}$.
   Solution: For the matrix A, such that $A\mathbf{v} = T(\mathbf{v})$ we obtain

   $$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

   We have $A\mathbf{v} = T(\mathbf{v}) = 3\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^2$.

   b) $S : \mathbb{R}^2 \to \mathbb{R}^2$, where S is a reflection with respect to the line $L = \{c \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ where } c \in \mathbb{R}\}$.
   Solution: For the matrix B, such that $B\mathbf{v} = S(\mathbf{v})$ we obtain

   $$B = [S(\mathbf{e}_1), S(\mathbf{e}_2)] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

   As this is a reflection we have for any vector $\mathbf{v}$ on the line L that $B\mathbf{v} = \mathbf{v}$. Furthermore, for any vector $\mathbf{w}$ attached at $\mathbf{0}$, orthogonal to the line L we have $B\mathbf{w} = -\mathbf{w}$.
   So we could take

   $$\text{Eigenvalue: } \lambda_1 = 1 \text{ , eigenvector: } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   $$\text{Eigenvalue: } \lambda_2 = -1 \text{ , eigenvector: } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   c) A horizontal shear in $\mathbb{R}^2$.
   Solution: For the matrix C, such that $C\mathbf{v} = H(\mathbf{v})$ we obtain

   $$C = [H(\mathbf{e}_1), H(\mathbf{e}_2)] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{where} \quad k \neq 0.$$

   As the vectors on the x-axis remain fixed by the shear, we have for any such vector that $C\mathbf{v} = \mathbf{v}$. So we could take

   $$\text{Eigenvalue: } \lambda_1 = 1 \text{ , eigenvector: } \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   Conversely, any vector that does not lie on the x-axis will be sheared, so it can never be a multiple of itself under the transformation. So $\lambda_1 = 1$ is the only eigenvalue.

d) $R(\frac{\pi}{2}) : \mathbb{R}^2 \to \mathbb{R}^2$, where $R(\frac{\pi}{2})$ is the counterclockwise rotation about the origin with angle $\frac{\pi}{2}$.
Solution: For the matrix R, such that $R\mathbf{v} = R(\frac{\pi}{2})(\mathbf{v})$ we obtain

$$R = [R(\frac{\pi}{2})(\mathbf{e}_1), R(\frac{\pi}{2})(\mathbf{e}_2)] = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Geometrically it is clear that there can not be an eigenvector, as every vector is rotated away from the line it spans by the map. This can also be checked by a calculation.

2. Find the characteristic polynomial of the following matrices, then determine the eigenvalues and their multiplicity. Justify your answer.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 2 & 0 \\ 4 & 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & -4 & -2 & 4 \\ 0 & -2 & 9 & 2 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & -\frac{1}{2} & 4 \end{bmatrix}.$$

Solution:

- For the matrix A we get the eigenvalues $\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 4}$ The multiplicity of both eigenvalues is $\boxed{1}$:

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}\right) = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4).$$

- For the matrix B we get the eigenvalues $\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 4}$: As

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = PAP^{-1}$$

we know that A and B are similar. By the lecture we know that similar matrices have the same eigenvalues.

- For the matrix $C - \lambda I_3$ we take the co-factor expansion along the second row, which contains the most zeros.

$$\begin{aligned} \det(-\lambda I_3) &= \begin{vmatrix} -3-\lambda & 2 & 1 \\ 0 & 2-\lambda & 0 \\ 4 & 2 & -1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -3-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} \\ &= (2-\lambda)(\lambda + 2 + \sqrt{5})(\lambda + 2 - \sqrt{5}). \end{aligned}$$

So the eigenvalues are $\boxed{\lambda_1 = 2}$, $\boxed{\lambda_2 = -2 - \sqrt{5}}$ and $\boxed{\lambda_3 = -2 + \sqrt{5}}$, each with multiplicity $\boxed{1}$.

- For the matrix $D - \lambda I_4$ we take the co-factor expansion along the first and then the second column, which contain the most zeros.

$$\det(D - \lambda I_4) = \begin{vmatrix} -2-\lambda & -4 & -2 & 4 \\ 0 & -2-\lambda & 9 & 2 \\ 0 & 0 & 6-\lambda & 2 \\ 0 & 0 & -\frac{1}{2} & 4-\lambda \end{vmatrix} = (-2-\lambda)^2(\lambda-5)^2 = (2+\lambda)^2(\lambda-5)^2.$$

So the eigenvalues are $\boxed{\lambda_1 = -2}$, $\boxed{\lambda_2 = 5}$, each with multiplicity $\boxed{2}$.

3. Let A be the 2×2 matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

a) Find the eigenvalues of A.
Solution: We solve the equation $\det(A - \lambda I_2) = 0$ for λ. We obtain

$$\det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (\lambda - 5)(\lambda + 2).$$

So the eigenvalues are $\boxed{\lambda_1 = 5}$ and $\boxed{\lambda_2 = -2}$.

b) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.
Solution: For the eigevalues we find the corresponding eigenspaces:

$$\text{Nul}(A - 5I_2) = \text{Span}\{\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{=\mathbf{v}_1}\} \text{ and } \text{Nul}(A + 2I_2) = \text{Span}\{\underbrace{\begin{bmatrix} -3 \\ 4 \end{bmatrix}}_{=\mathbf{v}_2}\}.$$

So a basis consisting of eigenvectors of $\mathbb{R}^2$ is $B = \{\mathbf{v}_1, \mathbf{v}_2\}$.

c) Diagonalize the matrix A.
Solution: From the lecture we know that

$$A = P_B D P_B^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} \right).$$

d) Calculate A^7.
Solution: Using the diagonalization of A we obtain:

$$A^2 = AA = (P_B D P_B^{-1})(P_B D P_B^{-1}) = P_B D^2 P_B^{-1}.$$

Continuing like that we obtain

$$A^7 = P_B D^7 P_B^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5^7 & 0 \\ 0 & (-2)^7 \end{bmatrix} \left(\frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} \right).$$

e) Diagonalize the inverse matrix A^{-1} of A.
Solution: Again we use the decomposition from part c): $A = P_B D P_B^{-1}$, so

$$A^{-1} = ((P_B D)P_B^{-1})^{-1} = (P_B^{-1})^{-1}(P_B D)^{-1} = P_B D^{-1} P_B^{-1} = P_B \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} P_B^{-1}.$$

As D^{-1} is also a diagonal matrix, we obtain a diagonalization of A^{-1}.

Hint: For d) and e) you can use the information from c).

4. **(Fibonacci sequence) (C)** Let $(x_k)_{k\in\mathbb{N}}$ be the Fibonacci sequence given by the recursive formula

$$x_0 = 0\ ,\ \ x_1 = 1\ ,\ \ \boxed{x_{k+2} = x_{k+1} + x_k}\ \ \text{for all}\ \ k \geq 0.$$

So we obtain the sequence

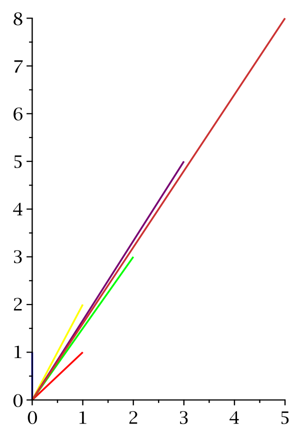
$$0\,,\,1\,,\,1\,,\,2\,,\,3\,,\,5\,,\,8\,,\,13\,,\,21\,,\,34\,,\,55\ldots$$

We have seen that the relation between the elements of the sequence can be described using a matrix:

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} \qquad (1)$$

b) In the vector form, this can be seen as a discrete dynamical system. Draw the first vectors $\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$, for $k = 0, 1, 2, 3, 4, 5, 6$ in the plane $\mathbb{R}^2$. What do you notice.

Solution: The direction of the vectors seems to converge to a certain direction. This would



mean that the ratio of consecutives numbers in the Fibonacci sequence converges. Indeed this ratio converges to the **golden ratio**

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$

c) Find the eigenvalues of A.
Solution: We solve the equation $\det(A - \lambda I_2) = 0$ for λ. We obtain

$$\det(A - \lambda I_2) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = (\lambda - \left(\frac{1+\sqrt{5}}{2}\right))(\lambda - \left(\frac{1-\sqrt{5}}{2}\right)).$$

So the eigenvalues are $\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2} = \varphi}$ and $\boxed{\lambda_2 = \frac{1-\sqrt{5}}{2} = \Phi}$.

d) Diagonalize the matrix A.
Solution: For the eigevalues we find the corresponding eigenspaces:

$$\text{Nul}(A - \varphi I_2) = \text{Span}\{\underbrace{\begin{bmatrix} \frac{1}{\varphi} \\ 1 \end{bmatrix}}_{=\mathbf{v}_1}\} \text{ and } \text{Nul}(A - \Phi I_2) = \text{Span}\{\underbrace{\begin{bmatrix} \frac{1}{\Phi} \\ 1 \end{bmatrix}}_{=\mathbf{v}_2}\}.$$

So a basis consisting of eigenvectors of $\mathbb{R}^2$ is $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Therefore we get for the diagonalization:

$$A = \begin{bmatrix} \frac{1}{\varphi} & \frac{1}{\Phi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \Phi \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{\Phi} \\ -1 & \frac{1}{\varphi} \end{bmatrix} \right).$$

e) It follows from (1) that $\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A^k \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. Use part d) to find a simple formula for A^k and then find a formula for x_k.
Solution: It follows from d) that

$$A = \begin{bmatrix} \frac{1}{\varphi} & \frac{1}{\Phi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^k & 0 \\ 0 & \Phi^k \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{\Phi} \\ -1 & \frac{1}{\varphi} \end{bmatrix} \right).$$

Therefore

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A^k \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\varphi} & \frac{1}{\Phi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^k & 0 \\ 0 & \Phi^k \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{\Phi} \\ -1 & \frac{1}{\varphi} \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^k - \Phi^k \\ \varphi^{k+1} - \Phi^{k+1} \end{bmatrix}.$$

Especially

$$\boxed{x_k = \frac{1}{\sqrt{5}} \left(\varphi^k - \Phi^k \right)}.$$

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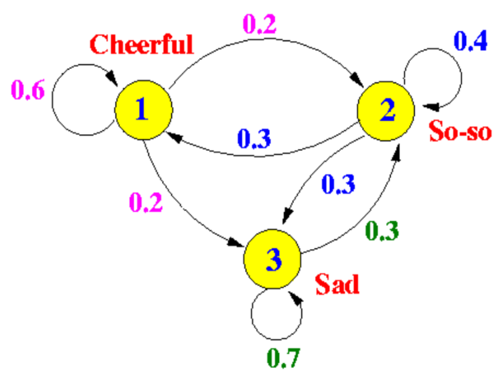
5. **(Markov chains) (C)** Draw your own (imaginary) mood diagram (see **Lecture 28**) and find the steady state.
Solution: We use the network described in **Lecture 28**:



We first construct the exchange table for the different states:

To \ From	Cheerful	So-so	Sad
Cheerful	0.6	0.3	0
So-so	0.2	0.4	0.3
Sad	0.2	0.3	0.7

So the corresponding stochastic matrix P is

$$P = \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.7 \end{bmatrix}.$$

Given a certain state vector $\mathbf{v}_0$ at time $t = 0$, the system evolves into the next round $t = 1$ by taking $\mathbf{v}_1 = P\mathbf{v}_0$. For $t = 2$, we have $\mathbf{v}_2 = P\mathbf{v}_1 = PP\mathbf{v}_0 = P^2\mathbf{v}_0$. In general we have $\boxed{\mathbf{v}_k = P^k\mathbf{v}_0}$. To find the steady state we have to solve the eigenvector-eigenvalue equation $P\mathbf{v} = \mathbf{v}$ or $(P - I_3)\mathbf{v} = \mathbf{0}$. Using a computer alegbra program we find:

$$\mathrm{Nul}(P - I_3) = \mathrm{Span}(\underbrace{\begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \\ 1 \end{bmatrix}}_{=\mathbf{v}}) \;\text{ or }\; \mathrm{Nul}(P - I_3) = \mathrm{Span}(\mathbf{q}) \;\text{ where }\; \mathbf{q} = \begin{bmatrix} 0.23 \\ 0.31 \\ 0.46 \end{bmatrix}.$$

To obtain the above probability vector $\mathbf{q}$ we have to scale the entries of $\mathbf{v}$, such that the sum is one. To this end we divide by the sum of the entries, which is $\frac{13}{6} \simeq 2.16$.
So, in this example, once the equilibrium is reached, the person is cheerful with a probability of 23 %, so-so with a probability of 31% and sad with a probability of 46 %.