**GENERAL INFORMATION**

- **Office hours:** Tu 1-3 pm, Th, Sun 2-4 pm in KH 229

- **Tutorial:** Tu, Th, Sun 7-9 pm in KH 105

- **Homework 5:** due next Wednesday at 4 pm outside KH 008. Please divide into the parts A, B, C and D and write your name on each part.
4 Vector Spaces

4.3 LINEAR INDEPENDENCE AND BASES
Summary:

1.) A **linearly independent spanning set** is called a **basis**.
2.) We can **find** a **basis** by **eliminating** vectors from a Span or by using the **row reduction algorithm**.
GEOMETRIC INTERPRETATION
**Definition:** An indexed set of vectors \( \{v_1, \ldots, v_p\} \) in \( V \) is said to be **linearly independent** if the vector equation

\[
x_1 v_1 + x_2 v_2 + \ldots + x_p v_p = 0
\]

has only the **trivial** solution \( x_1 = x_2 = \ldots = x_p = 0 \).

Otherwise the set \( \{v_1, \ldots, v_p\} \) is said to be **linearly dependent**. This means that there exist weights \( c_1, \ldots, c_p, \) **not all zero**, such that

\[
c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0 \quad (1)
\]

Equation (1) is called a **linear dependence relation** among \( v_1, \ldots, v_p \) when the weights are not all zero.
Theorem: The set of vectors \( \{v_1, \ldots, v_p\} \) in \( V \) is linearly independent if and only if any vector \( v \) in \( \text{Span} \{v_1, \ldots, v_p\} \) has a unique description as linear combination of the \( v_1, \ldots, v_p \).

Proof:
Bases

As this is a very practical property we define:

**Definition 1:** Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in $V$. Then a subset $B$ of vectors of $S$ is called a **basis** of $\text{Span}\{v_1, \ldots, v_p\}$, if the vectors in $B$ are **linearly independent** but still span $\text{Span}\{v_1, \ldots, v_p\}$.

**Definition 2:** Let $H$ be a subspace of a vector space $V$ and $B = \{b_1, \ldots, b_m\}$ be a subset of $H$. Then $B$ is a **basis** of $H$ if

i.) $B$ spans $H$, i.e. $\text{Span}\{b_1, \ldots, b_m\} = H$ 

ii.) The vectors $\{b_1, \ldots, b_m\}$ are **linearly independent**.

**Note:** This means that every vector in $H$ can be expressed uniquely as a **linear combination** of vectors in $B$.
- **Example:** Let $e_1, \ldots, e_n$ be the columns of the $n \times n$ matrix, $I_n$.
- That is,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- The set \{e_1, \ldots, e_n\} is called the **standard basis** for $\mathbb{R}^n$.
**Example:** Find a simple basis for

1.) the set of 2x2 matrices
2.) the set of polynomials $\mathbb{P}_3 \cong \mathbb{R}^4$ of degree smaller or equal to 3.
3.) the set of polynomials $\mathbb{P}_2 \cong \mathbb{R}^3$ of degree smaller or equal to 2.
4.) Check if the polynomials $B = \{1 + x, x + x^2, x^2 + 1\}$ form a basis of $\mathbb{P}_2$. 
HOW TO FIND A BASIS FOR A SPAN

We know:

- **Theorem**: (Characterization of Linearly Dependent Sets)
  An indexed set \( S = \{v_1, \ldots, v_p\} \) of two or more vectors in \( V \) is **linearly dependent** if and only if at least one of the vectors in \( S \) is a linear combination of the others.

**Proof**: As in Lecture 6, Theorem 7.

**Question**: If \( \{v_1, \ldots, v_p\} \) is linearly dependent, **can we remove a vector** \( v_j \) **such that**

\[
\text{Span}\{v_1, \ldots, v_p\} = \text{Span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_p\}?
\]
**Solution:** We can remove a vector $v_j$ that has a non-zero weight in the linear dependence relation.

**Consequence:** We can find a basis by successively removing vectors from a spanning set.

By the previous theorem the above solution is equal to

**Theorem 5: (Spanning set theorem)** Let $S=\{v_1, \ldots, v_p\}$ be a set in $V$, and let $H=\text{Span}\{v_1, \ldots, v_p\}$. If $v_j$ is a linear combination of the remaining vectors in $S$, then the set formed from $S$ by removing $v_j$ still spans $H$.

**Proof:** as in Lecture 6, Theorem 7. This proof can be directly translated into general vector spaces.
HOW TO FIND A BASIS FOR A SPAN

Example: Let \( v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \)

and \( H = \text{Span}\{v_1, v_2, v_3\}. \) Note that \( v_3 = 5v_1 + 3v_2. \)

1.) Show that \( \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2\}. \)

2.) Find a basis for the subspace \( H. \)

Solution:
In the case where \( \{v_1, \ldots, v_p\} \) are vectors in \( \mathbb{R}^m \), we can immediately identify a basis using the **row reduction algorithm**:

**Theorem:** Let \( S=\{v_1, \ldots, v_p\} \) be a set of vectors in \( \mathbb{R}^m \) and let \( A= [v_1, \ldots, v_p] \) be the matrix, whose columns are the \( v_j \). Then the vectors which form the **pivot columns of** \( A \) are form a **basis** of \( \text{Span}\{v_1, \ldots, v_p\} \).

**Proof:**
Translating this theorem into matrix notation we get:

- **Theorem 6**: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the corresponding standard matrix. Then the **pivot columns of $A$** form a **basis** for
  
  $T(\mathbb{R}^n) = \text{Col } A = \{b \in \mathbb{R}^m, \text{ s. t. } Ax = b \text{ for some } x \in \mathbb{R}^n \}$.

- **Warning**: The pivot columns of a matrix $A$ can only be read from the echelon form $U$ of $A$. But be careful to use the **pivot columns of $A$ itself** for the **basis** of $\text{Col } A$.
We have seen in Lecture 14:

- **Theorem:** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the corresponding standard matrix. If

  \[ Nul(T) = Nul A = \{ x \in \mathbb{R}^n, A x = 0 \} \]

  contains nonzero vectors then a **basis for $Nul A$** consist out of $q$ vectors, where $q$ equals the **number of non-pivot columns of $A$**.

- **Reminder:** We can find $Nul A$ explicitly by solving the homogeneous system of linear equations $A x = 0$.

- **Surprise:**
Example:

1.) Find a basis for $\text{Span}\{b_1, \ldots, b_5\}$, where

$$B = \begin{bmatrix} b_1 & b_2 & \ldots & b_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.) Find a basis for $\text{Nul} \ B$. 
Two Views of a Basis

When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.

If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span $V$.

View 1: A basis is a spanning set that is as small as possible.
View 2: A basis is also a **linearly independent** set that is **as large as possible**.

- If $S$ is a basis for $V$, and if $S$ is enlarged by one vector—say, $w$—from $V$, then the new set cannot be linearly independent, because $S$ spans $V$, and $w$ is therefore a linear combination of the elements in $S$. 