Math 22 –
Linear Algebra and its applications

- Lecture 16 -

Instructor: Bjoern Muetzel

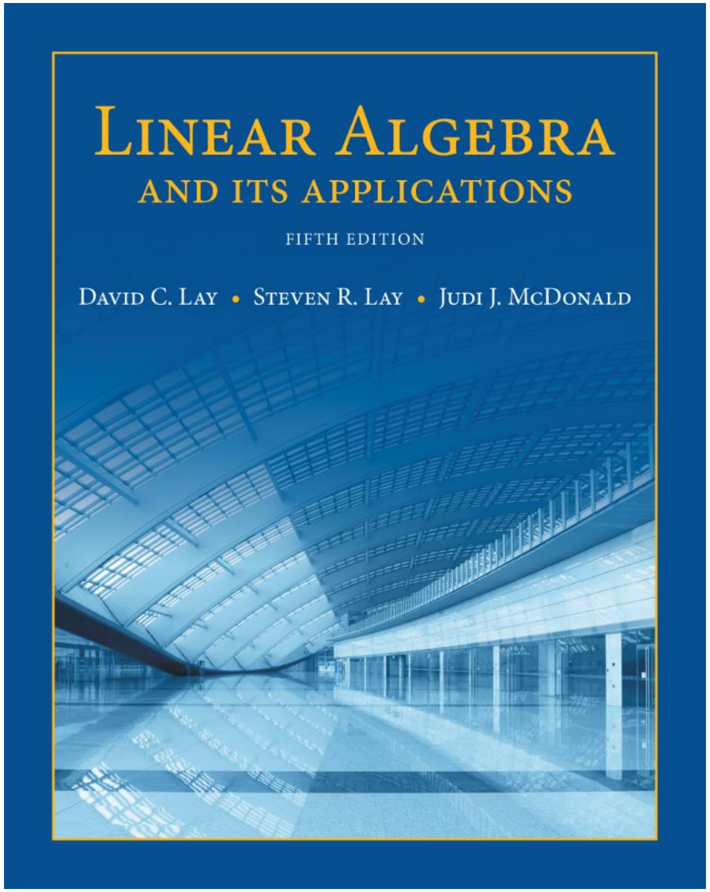
GENERAL INFORMATION

- **Office hours:** **Tu 1-3 pm**, Th, Sun 2-4 pm in **KH 229**
- **Tutorial:** **Tu**, Th, Sun **7-9 pm** in **KH 105**
- **Homework 5:** due **next Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.
- **Project:** Meeting next **weekend**!.

4 Vector Spaces

4.4

COORDINATE SYSTEMS



- <u>**Summary:**</u>

  1.) Using a basis we can define **coordinates** for a vector space V

  2.) If V has **n basis vectors** then it is isomorphic to $\mathbb{R}^n$

  3.) This means we can perform **all calculations in** $\mathbb{R}^{\boldsymbol{n}}$ and then **translate back** into V

GEOMETRIC INTERPRETATION

COORDINATE MAPS

- **Theorem 7 (Unique Representation Theorem):**
  Let B = {b_1, …, b_n} be a **basis** for vector space V. Then for each **x** in V, there exists a **unique** set of scalars c_1, …, c_n such that
  $$\boxed{x = c_1 b_1 + \cdots + c_n b_n} \quad (1)$$

- **Proof:** We have seen this in **Lecture 15**.

- **Definition:** Suppose B = {b_1, …, b_n} is a basis for V and **x** is in V. **The coordinates of x relative to the basis B** (or the **B-coordinates of x**) are the **weights** c_1, …, c_n such that
  $$\boxed{x = c_1 b_1 + \cdots + c_n b_n.}$$

COORDINATE MAPS

- If $c_1, \ldots, c_n$ are the **B**-coordinates of **x**, then the vector in $\mathbb{R}^n$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [x]_B = T_B(x)$$

is the **coordinate vector of x** or the **B-coordinate vector of x**.

- The mapping $[\,\cdot\,]_B = T_B : V \to \mathbb{R}^n, \mathrm{x} \mapsto T_B(x) = [x]_B$ is the **coordinate mapping (determined by B)**.

COORDINATE MAPS

- **Theorem 8:** Let B = {b_1, …, b_n} be a basis for a vector space *V*. Then the coordinate mapping

$$T_B: V \to \mathbb{R}^n, \mathrm{x} \mapsto T_B(x) = [x]_B$$

is a linear transformation that is both one-to-one and onto.

- **Proof:** Take two arbitrary vectors in *V*, say

$$\mathrm{u} = c_1 b_1 + \cdots + c_n b_n$$
$$\mathrm{w} = d_1 b_1 + \cdots + d_n b_n$$

Then $\mathrm{u} + \mathrm{w} =$

- It follows that

$$[\mathrm{u}+\mathrm{w}]_{\mathrm{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathrm{u}]_{\mathrm{B}} + [\mathrm{w}]_{\mathrm{B}}$$

- Hence

1.) $[\,u + w]_B = T_B(u+w) = T_B(\mathrm{u}) + T_B(\mathrm{v}) = [\,u]_B + [\,w]_B$.

So the coordinate mapping preserves addition.

- In a similar fashion for any a in $\mathbb{R}$ we have

2.) $[\,cu]_B = T_B(cu) = cT_B(\mathrm{u}) = c[\,u]_B$.

But 1.) and 2.) imply that $\boldsymbol{T_B}$ **is a linear map**.

T_B is both one-to-one and onto by the **Unique Representation Theorem.**

COORDINATE MAPS

- **Note:** The linearity of the coordinate mapping extends to linear combinations.
- If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then

$$\left[c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p\right]_{\mathrm{B}} = c_1\left[\mathrm{u}_1\right]_{\mathrm{B}} + \ldots + c_p\left[\mathrm{u}_p\right]_{\mathrm{B}}$$

- **<u>Definition:</u>** Let $T: V \rightarrow W$ be a linear transformation between vector spaces V and W. If T is both one to one and onto then T is called an **isomorphism** from V onto W.

In this case we say that V is **isomorphic** to W and write

$$V \cong W.$$

$V \cong \mathbb{R}^n$

Note: Using the coordinate map from **Theorem 8** we see that if V has a basis of **n** vectors then V is isomorphic to $\mathbb{R}^n$.

We will see in the following lecture that the **number of basis vectors** of V is an **invariant**. It is the dimension **dim(*V*)** of ***V*.**

Hence any finite dimensional vector space is isomorphic to some $\mathbb{R}^n$.

Consequence:

We can use a coordinate map to map any finite dimensional vector space V to some $\mathbb{R}^n$.

As the linearity of the coordinate map extends to linear combinations we can then **perform all vector calculations** in $\mathbb{R}^n$ and then **translate** our result **back to *V***.

$\mathrm{V} \cong \mathbb{R}^n$

- **Example:** The matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a standard basis **E** of the vector space of 2x2 matrices. Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} -3 & 4 \\ 0 & 0 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 1 & 4 \\ -2 & 2 \end{bmatrix}, D = \begin{bmatrix} 0 & 8 \\ 3 & -1 \end{bmatrix}.$$

1.) Write down $[\,A]_E$, $[\,B]_E$, $[\,C]_E$ and $[\,D]_E$

2.) Determine whether $\{A,B,C,D\}$ is a linearly independent set.

CHANGE OF BASIS IN $\mathbb{R}^n$

Given a concrete basis B = $\{b_1, \ldots, b_n\}$ for $\mathbb{R}^n$ and the usual standard basis E=$\{e_1, \ldots, e_n\}$.

How can we obtain the coordinate description of a vector in terms of B given in terms of E and vice versa?

1.) <u>One way is easy:</u> If u $= c_1 b_1 + \cdots + c_n b_n$, then $[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$.

Let $P_B = [b_1, b_2, \ldots, b_n]$ be the matrix whose columns are the b_i then the equation for u reads

$$\boxed{u = [u]_E = [b_1, b_2, \ldots, b_n]\ [u]_B = P_B [u]_B}$$

Since the columns of P_{B} form a basis for $\mathbb{R}^n$, P_{B} is invertible by the **Invertible Matrix Theorem**.

2.) Now the other direction is clear, too. As

$$u = [u]_E = P_B[u]_B$$

we can multiply both sides of the equation by P_B^{-1} . We get

$$P_B^{-1}u = P_B^{-1}[u]_E = P_B^{-1}P_B[u]_B = I_n[u]_B = [u]_B.$$

- **Definition:** P_{B} is called the **change-of-coordinates matrix** from B to the standard basis E in $\mathbb{R}^n$. Then for any u in $\mathbb{R}^n$

$$\boxed{u = [u]_E = P_B[u]_B} \text{ and } \boxed{[u]_B = P_B^{-1}[u]_E = P_B^{-1}u}$$

and therefore P_B^{-1} is a **change-of-coordinate matrix** from E to B.

CHANGE OF BASIS IN $\mathbb{R}^n$

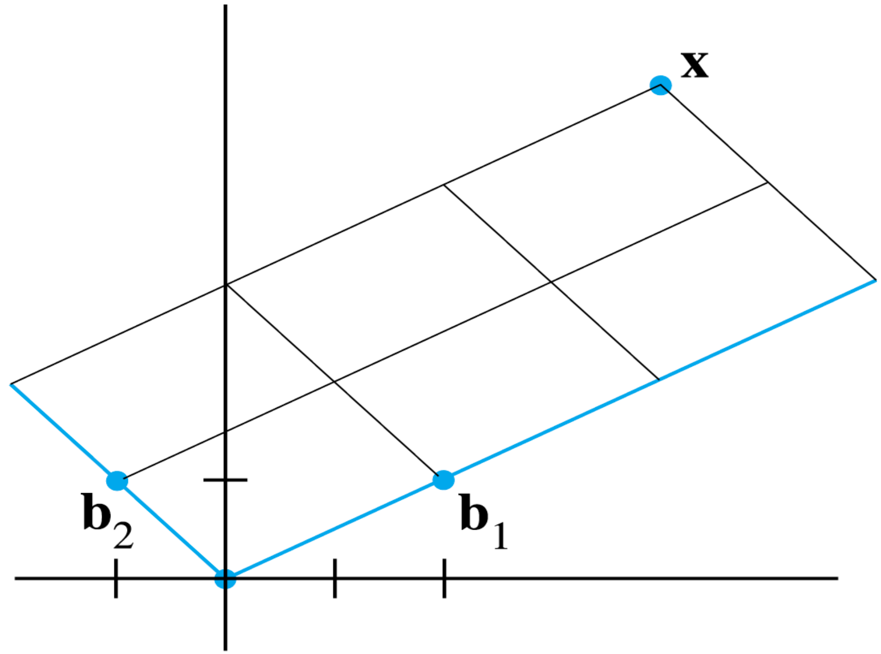
- **Example:** Let B ={b_1, b_2}, where $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

1.) Consider the vector $u = 3b_1 + 4b_2$. Write down $[u]_B$, then calculate $u = [u]_E$

2.) Write down the coordinate-change-matrix P_B and then calculate P_B^{-1}

3.) Find the coordinate vector $[x]_B$ of x relative to B, for $x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

CHANGE OF BASIS IN $\mathbb{R}^n$



The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)$.

- The matrix P_B changes the B-coordinates of a vector into the standard coordinates.